

Analysis Of A Tandem Queue With Bernoulli Routing And Load-Dependent Service Mechanism

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ABSTRACT

Queueing theory provides an essential framework for analyzing congestion and resource utilization in multi-stage service systems. This study examines a tandem queueing model incorporating Bernoulli routing and load-dependent service mechanisms, which better represent practical scenarios where customer flow is probabilistic and service rates vary with system load.

Customers arrive according to a stochastic process, and after service at the first node, they proceed to the next node with probability θ or leave the system with probability $1-\theta$. The service rates at each node depend on the number of customers present.

Key performance measures, including expected queue length, waiting time, and system throughput, are derived. A sensitivity analysis is also conducted to evaluate the impact of routing probabilities and load-dependent service parameters on system performance.

The results demonstrate that probabilistic routing and state-dependent service significantly influence system behaviour, making the proposed model more suitable for analyzing real-world multi-stage service systems.

KEYWORDS: Tandem queue; Bernoulli routing; Load-dependent service; Poisson process; Steady-state analysis; Performance measures.

1 INTRODUCTION

Queueing models play a vital role in analyzing congestion and improving service efficiency in a wide range of real-world systems, including communication networks, manufacturing processes, transportation systems, healthcare services, and computer networks. These models provide a mathematical framework to evaluate system performance, optimize resource utilization, and understand the behaviour of complex service mechanisms under varying conditions. In particular, tandem queueing systems are widely used to represent multi-stage service processes, where customers or jobs pass through a sequence of service nodes.

Early studies on tandem queueing systems primarily assumed deterministic routing, where customers move sequentially from one service node to the next without deviation. Such models, often developed under the assumption of stationary arrival and service processes, offer analytical simplicity but may not adequately capture the variability and uncertainty observed in real-world systems. In practical scenarios, customer movement between service stages is not always fixed; instead, it may depend on system conditions, service outcomes, or user decisions.

In recent years, there has been increasing interest in developing queueing models that incorporate more realistic system characteristics. One such feature is probabilistic routing, where customers are directed to subsequent service nodes or leave the system based on certain probabilities. This behaviour can be effectively modelled using Bernoulli routing mechanisms, which provide flexibility in representing systems with partial service completion, customer dropout, or dynamic routing decisions. Such routing structures are particularly relevant in applications such as healthcare systems, telecommunication networks, and service platforms, where not all customers proceed through every stage of service.

Another important characteristic of modern service systems is the dependence of service rates on the current system load. In many practical applications, service efficiency varies with the number of customers present in the system, leading to load-dependent service mechanisms. This feature captures congestion-sensitive behaviour and is essential for accurately modelling systems such as cloud computing environments, manufacturing lines, and traffic systems, where service capacity adapts dynamically to demand.

Our earlier work on three-node tandem queueing systems focused on models with deterministic sequential flow, where customers pass through all service nodes without the possibility of exiting the system at intermediate stages. While this framework provided valuable insights into multi-stage service dynamics with load-dependent service mechanisms, it does not account for scenarios where customers may leave the system after receiving partial service.

Despite the substantial body of literature on tandem queueing systems and load-dependent service mechanisms, relatively limited attention has been given to models that simultaneously incorporate probabilistic routing and load-dependent service rates. In particular, the interaction between Bernoulli routing and state-dependent service dynamics in multi-stage queueing systems remains insufficiently explored. This represents a significant gap in the existing literature, as many real-world systems exhibit both routing uncertainty and congestion-dependent service behaviour.

Motivated by these considerations, the present study develops and analyses a tandem queueing model that integrates Bernoulli routing with load-dependent service mechanisms. In the proposed system, customers arrive according to a stochastic process and receive service at the first node. Upon completion of service, a customer either proceeds to the next node with probability θ or leaves the system with probability $(1-\theta)$. The service rates at each node depend on the number of customers present, reflecting load-sensitive system dynamics.

The primary objective of this work is to derive the steady-state behaviour and key performance measures of the system, including expected queue length, waiting time, and system throughput. The proposed model provides a more realistic and flexible framework for analyzing multi-stage service systems and offers deeper insights into the impact of probabilistic routing and load-dependent service mechanisms on system performance.

The present study extends the existing framework to a tandem queueing system with probabilistic routing. In this model, customers arrive at the first service node according to a Poisson process with rate λ . After completing service at the first node, a customer proceeds to the next node with probability θ or leaves the system with probability $(1-\theta)$. Unlike classical tandem systems with fixed sequential flow, this structure captures more realistic service dynamics. The service rates at each node depend on the number of customers present, reflecting a load-dependent service mechanism.

The analytical framework of the model is developed using difference-differential equations, which describe the evolution of the system state over time. Key performance measures are derived, including the probability of an empty system, the expected number of customers in the system, server utilization, and average waiting time. In addition, system throughput, the variance of the queue length, and the coefficient of variation are also evaluated to provide a comprehensive understanding of system behaviour.

The transient analysis of the system is carried out through the derivation of the joint probability generating function (PGF) for the number of customers at each node. This approach enables the evaluation of performance measures at any given time and provides explicit expressions that reveal the influence of arrival rates, routing probabilities, and load-dependent service parameters on system performance.

Furthermore, a sensitivity analysis is performed using numerical examples to examine the impact of key system parameters. This analysis highlights how variations in routing probability and service characteristics affect overall system efficiency. A comparative study is also included to demonstrate the advantages of the proposed model over traditional tandem systems with deterministic routing, emphasizing the role of probabilistic routing and load-dependent service mechanisms in capturing realistic system behaviour.

There is relatively limited literature addressing tandem queueing systems that incorporate both probabilistic routing and load-dependent service mechanisms, particularly in multi-stage configurations. While classical models often assume fixed routing and constant service rates, such

assumptions may not adequately represent real-world systems where customer flow is uncertain and service efficiency varies with system load. The integration of Bernoulli routing allows customers to either proceed to subsequent service stages or exit the system, thereby introducing greater flexibility and realism into the modeling framework.

This paper focuses on a tandem queueing system in which customers arrive according to a Poisson process and are served at sequential nodes with load-dependent service rates. After receiving service at a node, customers are routed to the next node with a specified probability or leave the system, reflecting practical service dynamics. The proposed model captures both routing uncertainty and congestion-dependent service behaviour.

The organization of the paper is as follows. Section 2 presents the model assumptions, formulates the governing difference-differential equations, and develops the transient-state analysis. Section 3 derives the key performance measures, including average queue lengths, system throughput, waiting times, and the coefficient of variation of queue sizes. Section 4 provides numerical illustrations to demonstrate the analytical results. Section 5 presents a sensitivity analysis to examine the impact of system parameters, particularly routing probabilities and load-dependent service rates. Section 6 offers a comparison between the proposed model and classical tandem systems with deterministic routing. Finally, Section 7 concludes the paper with key findings and directions for future research.

2 QUEUEING MODEL

This Segment outlines the formulation of the proposed queueing model. We consider a tandem queueing system comprising three service nodes arranged sequentially. In this setup, the output of each queue acts as the input to the next, forming a linear flow of Users or jobs through the system. The aim is to model systems where the service dynamics are time-dependent and better represented by NHPP.

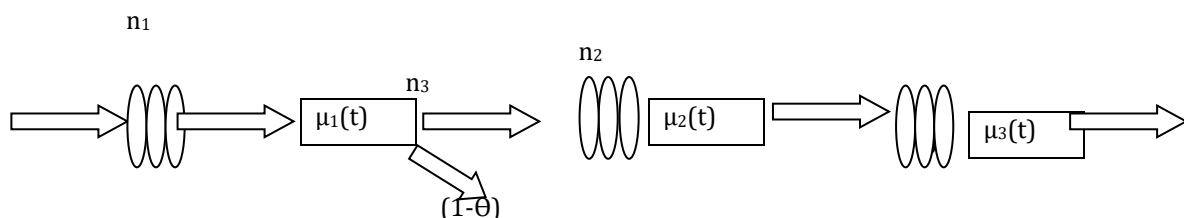
To develop the model, the following assumptions are made:

1. Service Mechanism: Each of the three service stations has a time-dependent service rate, modeled as a linear function of 't', i.e., $\mu_i(t) = \alpha_i + \beta_i(t)$, for $i=1,2,3$, where $\alpha_i, \beta_i \geq 0$.
2. Queue Discipline: The system follows the FCFS discipline at each node.
3. Interdependence: The User must complete service at one node before proceeding to the next; transitions between nodes are instantaneous and lossless.
4. System Capacity: Each queue has infinite capacity, ensuring no User is lost due to space limitations.
5. Initial Conditions: At $t=0$, the system is assumed to be empty.
6. After completing service from the first node the customer may join the second Queue with probability θ or leave the systems with probability $(1-\theta)$.

Based on these assumptions, a set of difference-differential equations is constructed to describe the time evolution of the system.

The schematic diagram illustrating the structure of the three-node tandem queueing system is presented in **Figure 1**. It visually represents the sequential flow of Users through the three service nodes, where each stage receives input from the previous one and passes its output to the next, forming a linear service pipeline.

Figure 1 schematic diagram of the three node TQM with first phase and time-dependent service rates.



Let $P_{n_1 n_2 n_3}(t)$ denote the probability that at time t , there are:

- n_1 Users in the 1st queue,
- n_2 Users in the 2nd queue, and

- n_3 Users in the 3rd queue.

The system dynamics are governed by a set of difference-differential equations that describe the evolution of these state probabilities over time, considering the arrival rate λ and the time-dependent service rates $\mu_1(t), \mu_2(t)$ and $\mu_3(t)$ for the three queues respectively.

These equations are derived based on the probability flow into and out of each state (n_1, n_2, n_3) , taking into account the possible transitions due to arrivals and service completions at each node.

Let $P_{n_1, n_2, n_3}(t)$ be the probability that there are n_1 customers in the first queue & n_2 customers in the second queue and n_3 customers in the third queue at time t . The difference-differential equations governing the model are:

$$\frac{\partial P_{n_1, n_2, n_3}(t)}{\partial t} = -(\lambda + n_1\mu_1(t) + n_2\mu_2(t) + n_3\mu_3(t)) P_{n_1, n_2, n_3}(t) + \lambda P_{n_1-1, n_2, n_3}(t) + (n_1 + 1)\mu_1(t) \Theta P_{n_1+1, n_2-1, n_3}(t) + (n_1 + 1)\mu_1(t) (1 - \Theta) P_{n_1+1, n_2, n_3}(t) + (n_2 + 1)\mu_2(t) P_{n_1, n_2+1, n_3-1}(t) + (n_3 + 1)\mu_3(t) P_{n_1, n_2, n_3+1}(t); \quad n_1, n_2, n_3 > 0$$

$$\frac{\partial P_{0, n_2, n_3}(t)}{\partial t} = -(\lambda + n_2\mu_2(t) + n_3\mu_3(t)) P_{0, n_2, n_3}(t) + \mu_1(t) \Theta P_{1, n_2-1, n_3}(t) + \mu_1(t) (1 - \Theta) P_{1, n_2, n_3}(t) + (n_2 + 1)\mu_2(t) P_{0, n_2+1, n_3-1}(t) + (n_3 + 1)\mu_3(t) P_{0, n_2, n_3+1}(t); \quad n_1 = 0 \text{ \& } n_2, n_3 > 0$$

$$\frac{\partial P_{n_1, 0, n_3}(t)}{\partial t} = -(\lambda + n_1\mu_1(t) + n_3\mu_3(t)) P_{n_1, 0, n_3}(t) + \lambda P_{n_1-1, 0, n_3}(t) + (n_1 + 1)\mu_1(t) (1 - \Theta) P_{n_1, 0, n_3}(t) + \mu_2(t) P_{n_1, 1, n_3-1}(t) + (n_3 + 1)\mu_3(t) P_{n_1, 0, n_3+1}(t); \quad n_2 = 0 \text{ \& } n_1, n_3 > 0$$

$$\frac{\partial P_{n_1, n_2, 0}(t)}{\partial t} = -(\lambda + n_1\mu_1(t) + n_2\mu_2(t)) P_{n_1, n_2, 0}(t) + \lambda P_{n_1-1, n_2, 0}(t) + (n_1 + 1)\mu_1(t) \Theta P_{n_1+1, n_2-1, 0}(t) + (n_1 + 1)\mu_1(t) (1 - \Theta) P_{n_1+1, n_2, 0}(t) + \mu_3(t) P_{n_1, n_2, 1}(t); \quad n_3 = 0 \text{ \& } n_1, n_2 > 0$$

$$\frac{\partial P_{0, 0, n_3}(t)}{\partial t} = -(\lambda + n_3\mu_3(t)) P_{0, 0, n_3}(t) + \mu_1(t) (1 - \Theta) P_{1, 0, n_3}(t) + \mu_2(t) P_{0, 1, n_3-1}(t) + (n_3 + 1)\mu_3(t) P_{0, 0, n_3+1}(t); \quad n_1 \text{ \& } n_2 = 0, n_3 > 0$$

$$\frac{\partial P_{n_1, 0, 0}(t)}{\partial t} = -(\lambda + n_1\mu_1(t)) P_{n_1, 0, 0}(t) + \lambda P_{n_1-1, 0, 0}(t) + (n_1 + 1)\mu_1(t) (1 - \Theta) P_{n_1+1, 0, 0}(t) + \mu_3(t) P_{n_1, 0, 1}(t); \quad n_1 = 0 \text{ \& } n_2, n_3 > 0$$

$$\frac{\partial P_{0, n_2, 0}(t)}{\partial t} = -(\lambda + n_2\mu_2(t)) P_{0, n_2, 0}(t) + \mu_1(t) \Theta P_{1, n_2-1, 0}(t) + \mu_1(t) (1 - \Theta) P_{1, n_2, 0}(t) + \mu_3(t) P_{0, n_2, 1}(t); \quad n_1, n_3 = 0 \text{ \& } n_2 > 0$$

$$\frac{\partial P_{0, 0, 0}(t)}{\partial t} = -(\lambda) P_{0, 0, 0}(t) + \mu_1(t) (1 - \Theta) P_{1, 0, 0}(t) + \mu_3(t) P_{0, 0, 1}(t); \quad n_1, n_2, n_3 = 0$$

The PGF of $P_{n_1, n_2, n_3}(t)$ is

$$P(S_1, S_2, S_3, t) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} P_{n_1, n_2, n_3}(t) S_1^{n_1} S_2^{n_2} S_3^{n_3} \quad (2)$$

Multiplying the Eq. (1) with $S_1^{n_1} S_2^{n_2} S_3^{n_3}$ and sum over all n_1, n_2, n_3 yields

$$\begin{aligned}
 \frac{\partial P_{n_1, n_2, n_3}(t)}{\partial t} S_1^{n_1} S_2^{n_2} S_3^{n_3} &= - \left(\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} (\lambda + n_1 \mu_1(t) + n_2 \mu_2(t) + n_3 \mu_3(t)) P_{n_1, n_2, n_3}(t) S_1^{n_1} S_2^{n_2} S_3^{n_3} \right. \\
 &+ \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \lambda P_{n_1-1, n_2, n_3}(t) S_1^{n_1} S_2^{n_2} S_3^{n_3} \\
 &+ \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} (n_1 + 1) \mu_1(t) \Theta P_{n_1+1, n_2-1, n_3}(t) S_1^{n_1} S_2^{n_2} S_3^{n_3} \\
 &+ \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} (n_1 + 1) \mu_1(t) (1 - \Theta) P_{n_1+1, n_2, n_3}(t) S_1^{n_1} S_2^{n_2} S_3^{n_3} \\
 &+ \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} (n_2 + 1) \mu_2(t) P_{n_1, n_2+1, n_3-1}(t) S_1^{n_1} S_2^{n_2} S_3^{n_3} \\
 &\left. + \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} (n_3 + 1) \mu_3(t) P_{n_1, n_2, n_3+1}(t) S_1^{n_1} S_2^{n_2} S_3^{n_3} \right) \quad (3)
 \end{aligned}$$

Upon simplification, we obtain

$$\begin{aligned}
 \frac{\partial P(S_1, S_2, S_3, t)}{\partial t} &= \mu_1(t)(1 - \Theta + \Theta S_2 - S_1) \frac{\partial P(S_1, S_2, S_3, t)}{\partial S_1} + \mu_2(t)(S_3 - S_2) \frac{\partial P(S_1, S_2, S_3, t)}{\partial S_2} \\
 &+ \mu_3(t) \frac{\partial P(S_1, S_2, S_3, t)}{\partial S_2} (1 - S_3) - \lambda(1 - S_1) P(S_1, S_2, S_3, t)
 \end{aligned} \quad (4)$$

Analysing the equation above equation by Using the method of Lagrange multipliers, the corresponding auxiliary equations are derived

$$\frac{\partial t}{1} = \frac{\partial S_1}{-\mu_1(t)(1 - \Theta + \Theta S_2 - S_1)} = \frac{\partial S_2}{-\mu_2(t)(S_3 - S_2)} = \frac{\partial S_3}{-\mu_3(t)(1 - S_3)} = \frac{\partial P}{-\lambda(1 - S_1)P(S_1, S_2, S_3, t)} \quad (5)$$

Let The service rates are time-dependent and linear, taking the form of

$$\mu_1(t) = \alpha_1 + \beta_1 t;$$

$$\mu_2(t) = \alpha_2 + \beta_2 t;$$

$$\mu_3(t) = \alpha_3 + \beta_3 t;$$

Analysing the 1st and 4th components from Equation (5), This leads to the following outcomes

$$a = (s_3 - 1) e^{\int \mu_3(t) dt} \quad (6)$$

Analysing the 1st and 3rd components from Equation (5), This leads to the following outcomes

$$b = s_2 e^{-\int \mu_2(t) dt} + (s_3 - 1) e^{-\int \mu_3(t) dt} \int \mu_2(t) e^{\int (\mu_3(t) - \mu_2(t)) dt} dt + \int \mu_2(t) e^{-\int \mu_2(t) dt} dt \quad (7)$$

Analysing the 1st and 2nd components from Equation (5), This leads to the following outcomes

$$\begin{aligned}
 c &= s_1 e^{-\int \mu_1(t) dt} + (\Theta s_2 e^{-\int \mu_2(t) dt} + \Theta(s_3 - 1) e^{-\int \mu_3(t) dt} \int \mu_2(t) e^{\int (\mu_3(t) - \mu_2(t)) dt} dt + \\
 &\int \Theta \mu_2(t) e^{-\int \mu_2(t) dt} dt) \left(\int \mu_1(t) e^{\int (\mu_2(t) - \mu_1(t)) dt} dt \right) - \left(\Theta(s_3 - 1) e^{-\int \mu_3(t) dt} \left(\int \mu_1(t) \left(\int \mu_2(t) e^{\int (\mu_3(t) - \mu_2(t)) dt} dt e^{\int \mu_2(t) dt} \right) e^{-\int \mu_1(t) dt} dt \right) - \right. \\
 &\left. \left(\int \Theta \mu_1(t) \left(\int \mu_2(t) e^{-\int \mu_2(t) dt} e^{\int \mu_2(t) dt} dt \right) e^{-\int \mu_1(t) dt} dt \right) \right) \quad (8)
 \end{aligned}$$

Analysing the 1st and 5th components from Equation (5), This leads to the following outcomes

$$\begin{aligned}
 d &= P(s_1, s_2, s_3, t) \exp \left[s_1 e^{-\int \mu_1(t) dt} + (\Theta s_2 e^{-\int \mu_2(t) dt} + \Theta(s_3 - 1) e^{-\int \mu_3(t) dt} \int \mu_2(t) e^{\int (\mu_3(t) - \mu_2(t)) dt} dt + \right. \\
 &\left. \int \Theta \mu_2(t) e^{-\int \mu_2(t) dt} dt) \left(\int \mu_1(t) e^{\int (\mu_2(t) - \mu_1(t)) dt} dt \right) - \right. \\
 &\left. \left(\Theta(s_3 - 1) e^{-\int \mu_3(t) dt} \left(\int \mu_1(t) \left(\int \mu_2(t) e^{\int (\mu_3(t) - \mu_2(t)) dt} dt e^{\int \mu_2(t) dt} \right) e^{-\int \mu_1(t) dt} dt \right) - \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & \left(\theta \int \mu_1(t) \left(\int \mu_2(t) e^{-\int \mu_2(t) dt} e^{\int \mu_2(t) dt} dt \right) e^{-\int \mu_1(t) dt} dt \right) * \left(\lambda e^{-\int \mu_1(t) dt} dt \right) + \left(\theta s_2 e^{-\int \mu_2(t) dt} + \right. \\
 & \theta(s_3 - 1) e^{-\int \mu_3(t) dt} \int \mu_2(t) e^{\int (\mu_3(t) - \mu_2(t)) dt} dt + \int \mu_2(t) e^{-\int \mu_2(t) dt} dt \left. \right) * \\
 & \left(\lambda e^{\int \mu_1(t) dt} \left(\int \mu_1(t) e^{\int (\mu_2(t) - \mu_1(t)) dt} dt \right) dt \right) - \theta(s_3 - 1) e^{-\int \mu_3(t) dt} \left(\int \lambda e^{\int \mu_1(t) dt} \left(\int \mu_1(t) \left(\int \mu_2(t) e^{\int (\mu_3(t) - \mu_2(t)) dt} e^{\int \mu_2(t) dt} e^{-\int \mu_1(t) dt} dt \right) dt \right) - \right. \\
 & \left. \left(\int \theta \lambda e^{\int \mu_1(t) dt} \left(\int \mu_1(t) \left(\int \mu_2(t) e^{-\int \mu_2(t) dt} e^{\int \mu_2(t) dt} dt \right) e^{-\int \mu_1(t) dt} dt \right) dt \right) + \int \lambda dt \right) \quad (9)
 \end{aligned}$$

Here, the parameters a, b, c and d are arbitrary constants that define the time-dependent service rates of the three service nodes, typically in the form $\mu_i(t) = \alpha_i + \beta_i(t)$, for $i=1,2,3$. These parameters allow flexibility in modeling the increasing of variable capacity of servers over time. The initial conditions for the system are specified as: $P_{0,0,0}(0)=1$, and $P_{n_1, n_2, n_3}(0)=0$ for all $(n_1, n_2, n_3) \neq (0,0,0)$, indicating that the system starts empty at time $t=0$.

To analyze the system, we define the PGF of the No. of Users in each queue at time t as:

$$\begin{aligned}
 P(S_1, S_2, S_3, t) &= \exp \left(-\frac{\lambda s_1 e^{-\int \mu_1(t) dt}}{\alpha_1} \right) - \\
 & \frac{\lambda \theta \left(s_2 e^{-\int \mu_2(t) dt} + (s_3 - 1) e^{-\int \mu_3(t) dt} \int \mu_2(t) e^{\int (\mu_3(t) - \mu_2(t)) dt} dt + \int \mu_2(t) e^{-\int \mu_2(t) dt} dt \right) \left(\int \mu_1(t) e^{\int (\mu_2(t) - \mu_1(t)) dt} dt \right)}{\alpha_1} + \\
 & \frac{\lambda \theta \left((s_3 - 1) e^{-\int \mu_3(t) dt} \left(\int \mu_1(t) \left(\int \mu_2(t) e^{\int (\mu_3(t) - \mu_2(t)) dt} e^{\int \mu_2(t) dt} e^{-\int \mu_1(t) dt} dt \right) dt \right) \right)}{\alpha_1} + \\
 & \frac{\lambda \theta \left(\int \mu_1(t) \left(\int \mu_2(t) e^{-\int \mu_2(t) dt} e^{\int \mu_2(t) dt} dt \right) e^{-\int \mu_1(t) dt} dt \right)}{\alpha_1} + \frac{\lambda \theta (s_2 e^{-\int \mu_2(t) dt})}{\alpha_2 - \alpha_1} + \\
 & \frac{\lambda \theta \left((s_3 - 1) e^{-\int \mu_3(t) dt} \int \mu_2(t) e^{\int (\mu_3(t) - \mu_2(t)) dt} dt \right)}{\alpha_2 - \alpha_1} + \frac{\lambda \theta \left(\int \mu_2(t) e^{-\int \mu_2(t) dt} dt \right)}{\alpha_2 - \alpha_1} - \frac{\lambda \theta (s_3 - 1) e^{-\int \mu_3(t) dt} * \alpha_2}{(\alpha_3 - \alpha_2)(\alpha_3 - \alpha_1)} - \\
 & \lambda \theta \left(\frac{s_2 e^{-\int \mu_2(t) dt}}{\alpha_2} \right) - \frac{\lambda \theta \left((s_3 - 1) e^{-\int \mu_3(t) dt} \int \mu_2(t) e^{\int (\mu_3(t) - \mu_2(t)) dt} dt \right)}{\alpha_2} - \frac{\lambda \theta \left(\int \mu_2(t) e^{-\int \mu_2(t) dt} dt \right)}{\alpha_2} - \\
 & \frac{\lambda \theta \left((s_3 - 1) e^{-\int \mu_3(t) dt} \right)}{\alpha_2} + \frac{\lambda \theta \left((s_3 - 1) e^{-\int \mu_3(t) dt} \right)}{\alpha_2} + \\
 & \left[s_1 e^{-\int \mu_1(t) dt} + \left(\theta s_2 e^{-\int \mu_2(t) dt} + \theta(s_3 - 1) e^{-\int \mu_3(t) dt} \int \mu_2(t) e^{\int (\mu_3(t) - \mu_2(t)) dt} dt + \theta \int \mu_2(t) e^{-\int \mu_2(t) dt} dt \right) \right. \\
 & \left. \left(\int \mu_1(t) e^{\int (\mu_2(t) - \mu_1(t)) dt} dt \right) - \right. \\
 & \left. \left(\theta(s_3 - 1) e^{-\int \mu_3(t) dt} \left(\int \mu_1(t) \left(\int \mu_2(t) e^{\int (\mu_3(t) - \mu_2(t)) dt} e^{\int \mu_2(t) dt} e^{-\int \mu_1(t) dt} dt \right) dt \right) - \right. \right. \\
 & \left. \left. \left(\theta \int \mu_1(t) \left(\int \mu_2(t) e^{-\int \mu_2(t) dt} e^{\int \mu_2(t) dt} dt \right) e^{-\int \mu_1(t) dt} dt \right) \right) \right. \\
 & \left. - \left(\theta s_2 e^{-\int \mu_2(t) dt} + \theta(s_3 - 1) e^{-\int \mu_3(t) dt} \int \mu_2(t) e^{\int (\mu_3(t) - \mu_2(t)) dt} dt + \right. \right. \\
 & \left. \left. \int \theta \mu_2(t) e^{-\int \mu_2(t) dt} dt \right) \left(\int \lambda e^{\int \mu_1(t) dt} \left(\int \mu_1(t) e^{\int (\mu_2(t) - \mu_1(t)) dt} dt \right) dt \right) + \left(\theta(s_3 - 1) e^{-\int \mu_3(t) dt} \left(\int \lambda e^{\int \mu_1(t) dt} \left(\int \mu_1(t) \left(\int \mu_2(t) e^{\int (\mu_3(t) - \mu_2(t)) dt} e^{\int \mu_2(t) dt} e^{-\int \mu_1(t) dt} dt \right) dt \right) + \right. \right. \right. \\
 & \left. \left. \left(\theta \int \lambda e^{\int \mu_1(t) dt} \left(\int \mu_1(t) \left(\int \mu_2(t) e^{-\int \mu_2(t) dt} e^{\int \mu_2(t) dt} dt \right) e^{-\int \mu_1(t) dt} dt \right) dt \right) - \int \lambda dt \right) \right) \quad (10)
 \end{aligned}$$

3. ATTRIBUTES OF THE QUEUING MODEL

By carrying out the expansion of the equation $P(s_1, s_2, s_3, t)$ as given in Eq. (10) and isolating the constant components, we derive the expression for the probability that there are no Users in the queue.

$$\begin{aligned}
 P_{000}(t) = \exp * (-\lambda) & \left\{ \left(e^{-(\alpha_1 t + \beta_1 \frac{t^2}{2})} \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv - \frac{1}{\alpha_1} \right) \right) \right. \\
 & + \Theta \left(e^{-(\alpha_2 t + \beta_2 \frac{t^2}{2})} \left(\frac{1}{(\alpha_2 - \alpha_1)} - \int_0^t \frac{(\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1) \frac{v^2}{2}} dv}{\alpha_1} \right) \right. \\
 & + e^{-(\alpha_2 t + \beta_2 \frac{t^2}{2})} \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv \int_0^t (\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1) \frac{v^2}{2}} dv \right. \\
 & - \left. \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} \int_0^t (\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1) \frac{v^2}{2}} dv \right) dv \right) - \frac{1}{\alpha_2} \Bigg) \\
 & + \Theta \left[e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \left(\int_0^t (\alpha_1 \right. \right. \\
 & + \beta_1 v) \left(\int_0^t (\alpha_2 + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2) \frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-(\alpha_2 v + \beta_2 \frac{v^2}{2})} dv \right. \\
 & - \left. \left(\frac{\alpha_1 \alpha_2}{\alpha_3 (\alpha_3 - \alpha_2) (\alpha_3 - \alpha_1)} \right) \right) \right. \\
 & + e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \int_0^t (\alpha_2 + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2) \frac{v^2}{2}} dv \left(\left(\frac{\alpha_1}{\alpha_2 (\alpha_2 - \alpha_1)} \right) \right. \\
 & - \left. \int_0^t \frac{(\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1) \frac{v^2}{2}} dv}{\alpha_1} \right) \\
 & + e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \int_0^t (\alpha_2 + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2) \frac{v^2}{2}} dv \left(\left(\int_0^t (\alpha_1 \right. \right. \\
 & + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1) \frac{v^2}{2}} dv \int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} dv \Bigg) \\
 & - \int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} \left(\int_0^t (\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1) \frac{v^2}{2}} dv \right) dv \Bigg) \\
 & + e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \left(\int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} \left(\int_0^t (\alpha_1 + \beta_1 v) \left(\int_0^t (\alpha_2 \right. \right. \right. \\
 & + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2) \frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-(\alpha_2 v + \beta_2 \frac{v^2}{2})} dv \Bigg) dv \Bigg) \\
 & - \left(\int_0^t (\alpha_1 + \beta_1 v) \left(\int_0^t (\alpha_2 \right. \right. \right. \\
 & + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2) \frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-(\alpha_2 v + \beta_2 \frac{v^2}{2})} dv \Bigg) \left(\int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} dv \right) \Bigg) \Bigg\}
 \end{aligned}$$

Taking $s_2=1, s_3=1$ in $P(s_1, s_2, s_3, t)$, we obtain the pgf of the 1st queue size as

$$P(s_1, t) = \exp \left(\lambda (s_1 - 1) e^{-(\alpha_1 t + \beta_1 \frac{t^2}{2})} \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv - \frac{1}{\alpha_1} \right) \right) ; \lambda < \alpha_1, \beta_1 \quad (12)$$

By developing $P(s_1, t)$ and gathering the constant components, we obtain the probability that the 1st queue is empty as

$$P_{0..}(t) = \exp \left(-\lambda e^{-(\alpha_1 t + \beta_1 \frac{t^2}{2})} \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv - \frac{1}{\alpha_1} \right) \right) \quad (13)$$

The avg. No. of Users in the 1st queue is

$$L_1(t) = \lambda e^{-(\alpha_1 t + \beta_1 \frac{t^2}{2})} \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv - \frac{1}{\alpha_1} \right) \quad (14)$$

The utilization of the 1st service station is

$$U_1(t) = 1 - \exp \left(-\lambda e^{-(\alpha_1 t + \beta_1 \frac{t^2}{2})} \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv - \frac{1}{\alpha_1} \right) \right) \quad (15)$$

The throughput of the 1st service station is

$$\text{ThP}_1(t) = (\alpha_1 + \beta_1 t) \left[1 - \exp \left(-\lambda e^{-(\alpha_1 t + \beta_1 \frac{t^2}{2})} \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv - \frac{1}{\alpha_1} \right) \right) \right] \quad (16)$$

The avg. waiting time of a User in the 1st queue is

$$W_1(t) = \frac{\lambda e^{-(\alpha_1 t + \beta_1 \frac{t^2}{2})} \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv - \frac{1}{\alpha_1} \right)}{(\alpha_1 + \beta_1 t) \left[1 - \exp \left(-\lambda e^{-(\alpha_1 t + \beta_1 \frac{t^2}{2})} \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv - \frac{1}{\alpha_1} \right) \right) \right]} \quad (17)$$

The var. of the No. of Users in the 1st queue is

$$V_1(t) = \lambda e^{-(\alpha_1 t + \beta_1 \frac{t^2}{2})} \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv - \frac{1}{\alpha_1} \right) \quad (18)$$

The CV of the No. of Users in the 1st system is

$$CV_1(t) = \left(\lambda e^{-(\alpha_1 t + \beta_1 \frac{t^2}{2})} \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv - \frac{1}{\alpha_1} \right) \right)^{-1/2} * 100 \quad (19)$$

Taking $s_1=1, s_3=1$ in $P(s_1, s_2, s_3, t)$, we obtain the pgf of the 2nd queue size as

$$\begin{aligned} P(s_2, t) = \exp. \lambda. \theta & \left((s_2 - 1) e^{-(\alpha_2 t + \beta_2 \frac{t^2}{2})} \left(\frac{1}{(\alpha_2 - \alpha_1)} - \int_0^t \frac{(\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv}{\alpha_1} \right) \right. \\ & + (s_2 - 1) e^{-(\alpha_2 t + \beta_2 \frac{t^2}{2})} \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv \int_0^t (\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv \right. \\ & \left. \left. - \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} \int_0^t (\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv \right) dv \right) - \frac{1}{\alpha_2} \right) ; \lambda \\ & < \alpha_1, \beta_1 \quad (20) \end{aligned}$$

By developing $P(s_2, t)$ and gathering the constant components, we obtain the probability that the 2nd queue is empty as

$$\begin{aligned} P_0(t) = -\exp. \lambda. \theta & \left(e^{-(\alpha_2 t + \beta_2 \frac{t^2}{2})} \left(\frac{1}{(\alpha_2 - \alpha_1)} - \int_0^t \frac{(\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv}{\alpha_1} \right) \right. \\ & + e^{-(\alpha_2 t + \beta_2 \frac{t^2}{2})} \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv \int_0^t (\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv \right. \\ & \left. \left. - \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} \int_0^t (\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv \right) dv \right) - \frac{1}{\alpha_2} \right) ; \lambda \\ & < \alpha_1, \beta_1 \quad (21) \end{aligned}$$

The avg. No. of Users in the 2nd queue is

$$L_2(t) = \lambda \cdot \theta \left(e^{-(\alpha_2 t + \beta_2 \frac{t^2}{2})} \left(\frac{1}{(\alpha_2 - \alpha_1)} - \int_0^t \frac{(\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv}{\alpha_1} \right) \right. \\ \left. + e^{-(\alpha_2 t + \beta_2 \frac{t^2}{2})} \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv \int_0^t (\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv \right. \right. \\ \left. \left. - \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} \int_0^t (\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv \right) dv \right) - \frac{1}{\alpha_2} \right) ; \lambda \\ < \alpha_1, \beta_1 \quad (22)$$

The utilization of the 2nd service station is

$$U_2(t) = 1 - \exp. \lambda \cdot \theta \left(-e^{-(\alpha_2 t + \beta_2 \frac{t^2}{2})} \left(\frac{1}{(\alpha_2 - \alpha_1)} - \int_0^t \frac{(\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv}{\alpha_1} \right) \right. \\ \left. + e^{-(\alpha_2 t + \beta_2 \frac{t^2}{2})} \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv \int_0^t (\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv \right. \right. \\ \left. \left. - \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} \int_0^t (\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv \right) dv \right) - \frac{1}{\alpha_2} \right) ; \lambda \\ < \alpha_1, \beta_1 \quad (23)$$

The throughput of the 2nd service station is

$$\text{ThP}_2(t) = (\alpha_1 + \beta_1 t) \left[1 \right. \\ \left. - \exp. \lambda \cdot \theta \left(e^{-(\alpha_2 t + \beta_2 \frac{t^2}{2})} \left(\frac{1}{(\alpha_2 - \alpha_1)} \right. \right. \right. \\ \left. \left. - \int_0^t \frac{(\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv}{\alpha_1} \right) \right. \\ \left. + e^{-(\alpha_2 t + \beta_2 \frac{t^2}{2})} \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv \int_0^t (\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv \right. \right. \\ \left. \left. - \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} \int_0^t (\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv \right) dv \right) \right. \\ \left. - \frac{1}{\alpha_2} \right) \quad (24)$$

The avg. waiting time of a User in the 2nd queue is

$$W_2(t) = \frac{L_2(t)}{\text{ThP}_2(t)} \quad (25)$$

The var. of the No. of Users in the 2nd queue is

$$V_2(t) = \lambda \cdot \Theta \left(e^{-\left(\alpha_2 t + \beta_2 \frac{t^2}{2}\right)} \left(\frac{1}{(\alpha_2 - \alpha_1)} - \int_0^t \frac{(\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1) \frac{v^2}{2}} dv}{\alpha_1} \right) + \right. \\ \left. e^{-\left(\alpha_2 t + \beta_2 \frac{t^2}{2}\right)} \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv \int_0^t (\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1) \frac{v^2}{2}} dv - \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv \int_0^t (\alpha_1 + \right. \right. \right. \\ \left. \left. \left. \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1) \frac{v^2}{2}} dv \right) - \frac{1}{\alpha_2} \right) \right) ; \lambda < \alpha_1, \beta_1 \quad (26)$$

The CVof the No. of Users in the 2nd system is

$$CV_2(t) = \left(\lambda \cdot \Theta \left(e^{-\left(\alpha_2 t + \beta_2 \frac{t^2}{2}\right)} \left(\frac{1}{(\alpha_2 - \alpha_1)} - \int_0^t \frac{(\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1) \frac{v^2}{2}} dv}{\alpha_1} \right) + \right. \right. \\ \left. \left. e^{-\left(\alpha_2 t + \beta_2 \frac{t^2}{2}\right)} \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv \int_0^t (\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1) \frac{v^2}{2}} dv - \left(\int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv \int_0^t (\alpha_1 + \right. \right. \right. \\ \left. \left. \left. \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1) \frac{v^2}{2}} dv \right) - \frac{1}{\alpha_2} \right) \right) \right)^{-1/2} \quad (27)$$

100

Taking $s_1=1, s_2=1$ in $P(s_1, s_2, s_3, t)$, we obtain the pgf of the 3rd queue size as

$$P(s_3, t) = \exp. \lambda \cdot \Theta \left[(s_3 - 1) e^{-\left(\alpha_3 t + \beta_3 \frac{t^2}{2}\right)} \left(\int_0^t (\alpha_1 + \beta_1 v) \left(\int_0^t (\alpha_2 + \right. \right. \right. \\ \left. \left. \left. \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2) \frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-\left(\alpha_2 v + \beta_2 \frac{v^2}{2}\right)} dv - \left(\frac{\alpha_1 \alpha_2}{\alpha_3 (\alpha_3 - \alpha_2) (\alpha_3 - \alpha_1)} \right) \right) + (s_3 - \right. \right. \\ \left. \left. 1) e^{-\left(\alpha_3 t + \beta_3 \frac{t^2}{2}\right)} \int_0^t (\alpha_2 + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2) \frac{v^2}{2}} dv \left(\left(\frac{\alpha_1}{\alpha_2 (\alpha_2 - \alpha_1)} \right) - \right. \right. \\ \left. \left. \int_0^t \frac{(\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1) \frac{v^2}{2}} dv}{\alpha_1} \right) + (s_3 - 1) e^{-\left(\alpha_3 t + \beta_3 \frac{t^2}{2}\right)} \int_0^t (\alpha_2 + \right. \\ \left. \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2) \frac{v^2}{2}} dv \left(\int_0^t (\alpha_1 + \right. \right. \\ \left. \left. \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1) \frac{v^2}{2}} dv \int_0^t e^{\left(\alpha_1 v + \beta_1 \frac{v^2}{2}\right)} dv - \int_0^t e^{\left(\alpha_1 v + \beta_1 \frac{v^2}{2}\right)} \left(\int_0^t (\alpha_1 + \right. \right. \\ \left. \left. \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1) \frac{v^2}{2}} dv \right) dv \right) + (s_3 - 1) e^{-\left(\alpha_3 t + \beta_3 \frac{t^2}{2}\right)} \left(\int_0^t e^{\left(\alpha_1 v + \beta_1 \frac{v^2}{2}\right)} \left(\int_0^t (\alpha_1 + \beta_1 v) \left(\int_0^t (\alpha_2 + \right. \right. \right. \\ \left. \left. \left. \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2) \frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-\left(\alpha_2 v + \beta_2 \frac{v^2}{2}\right)} dv \right) dv - \left(\int_0^t (\alpha_1 + \beta_1 v) \left(\int_0^t (\alpha_2 + \right. \right. \right. \\ \left. \left. \left. \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2) \frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-\left(\alpha_2 v + \beta_2 \frac{v^2}{2}\right)} dv \right) \left(\int_0^t e^{\left(\alpha_1 v + \beta_1 \frac{v^2}{2}\right)} dv \right) \right) \right] \quad (28)$$

By developing $P(s_3, t)$ and gathering the constant components, we obtain the probability that the 3rd queue is empty as

$$\begin{aligned}
& P_{..0}(t) \\
& = \exp.(-\lambda). \Theta \left[e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \left(\int_0^t (\alpha_1 \right. \right. \\
& + \beta_1 v) \left(\int_0^t (\alpha_2 + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} \right) e^{-(\alpha_2 v + \beta_2 \frac{v^2}{2})} dv \\
& - \left(\frac{\alpha_1 \alpha_2}{\alpha_3(\alpha_3 - \alpha_2)(\alpha_3 - \alpha_1)} \right) \Big) \\
& + e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \int_0^t (\alpha_2 + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv \left(\left(\frac{\alpha_1}{\alpha_2(\alpha_2 - \alpha_1)} \right) \right. \\
& - \left. \int_0^t \frac{(\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv}{\alpha_1} \right) \\
& + e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \int_0^t (\alpha_2 + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv \left(\left(\int_0^t (\alpha_1 \right. \right. \\
& + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv \int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} dv \\
& - \int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} \left(\int_0^t (\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv \right) dv \Big) + \\
& + e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \left(\int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} \left(\int_0^t (\alpha_1 + \beta_1 v) \left(\int_0^t (\alpha_2 \right. \right. \right. \\
& + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-(\alpha_2 v + \beta_2 \frac{v^2}{2})} dv \Big) dv \Big) \\
& - \left(\int_0^t (\alpha_1 + \beta_1 v) \left(\int_0^t (\alpha_2 \right. \right. \\
& + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-(\alpha_1 v + \beta_1 \frac{v^2}{2})} dv \Big) \left(\int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} dv \right) \Big) \Big] \quad (29)
\end{aligned}$$

The avg. No. of Users in the 3rd queue is

$$\begin{aligned}
L_3(t) & = \lambda. \Theta \left[e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \left(\int_0^t (\alpha_1 + \beta_1 v) \left(\int_0^t (\alpha_2 + \right. \right. \right. \\
& \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-(\alpha_2 v + \beta_2 \frac{v^2}{2})} dv - \left(\frac{\alpha_1 \alpha_2}{\alpha_3(\alpha_3 - \alpha_2)(\alpha_3 - \alpha_1)} \right) \Big) + e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \int_0^t (\alpha_2 + \\
& \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv \left(\left(\frac{\alpha_1}{\alpha_2(\alpha_2 - \alpha_1)} \right) - \int_0^t \frac{(\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv}{\alpha_1} \right) + e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \int_0^t (\alpha_2 + \\
& \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv \left(\left(\int_0^t (\alpha_1 + \right. \right. \\
& \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv \int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} dv - \int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} \left(\int_0^t (\alpha_1 + \right. \\
& \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv \Big) dv \Big) + e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \left(\int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} \left(\int_0^t (\alpha_1 + \beta_1 v) \left(\int_0^t (\alpha_2 + \right. \right. \right. \\
& \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-(\alpha_2 v + \beta_2 \frac{v^2}{2})} dv \Big) dv \Big) - \left(\int_0^t (\alpha_1 + \beta_1 v) \left(\int_0^t (\alpha_2 + \right. \right. \\
& \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-(\alpha_1 v + \beta_1 \frac{v^2}{2})} dv \Big) \left(\int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} dv \right) \Big) \Big] \quad (30)
\end{aligned}$$

The utilization of the 3rd service station is

$$\begin{aligned}
 U_3(t) = 1 - \exp.(-\lambda) \cdot \theta & \left[e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \left(\int_0^t (\alpha_1 \right. \right. \\
 & + \beta_1 v) \left(\int_0^t (\alpha_2 + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} \right) e^{-(\alpha_2 v + \beta_2 \frac{v^2}{2})} dv \\
 & - \left(\frac{\alpha_1 \alpha_2}{\alpha_3(\alpha_3 - \alpha_2)(\alpha_3 - \alpha_1)} \right) \Big) \\
 & + e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \int_0^t (\alpha_2 + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv \left(\left(\frac{\alpha_1}{\alpha_2(\alpha_2 - \alpha_1)} \right) \right. \\
 & - \left. \int_0^t \frac{(\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv}{\alpha_1} \right) \\
 & + e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \int_0^t (\alpha_2 + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv \left(\left(\int_0^t (\alpha_1 \right. \right. \\
 & + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv \int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} dv \\
 & - \int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} \left(\int_0^t (\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv \right) dv \\
 & + e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \left(\int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} \left(\int_0^t (\alpha_1 + \beta_1 v) \left(\int_0^t (\alpha_2 \right. \right. \right. \\
 & + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-(\alpha_2 v + \beta_2 \frac{v^2}{2})} dv \right) dv \\
 & - \left(\int_0^t (\alpha_1 + \beta_1 v) \left(\int_0^t (\alpha_2 \right. \right. \\
 & + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} \right) e^{-(\alpha_1 v + \beta_1 \frac{v^2}{2})} dv \left(\int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} dv \right) \Big) \Big]
 \end{aligned}
 \tag{31}$$

The throughput of the 3rd service station is

$$\begin{aligned}
 ThP_3(t) = (\alpha_3 + \beta_3 t) & \left\{ 1 - \exp.(-\lambda) \cdot \theta \left[e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \left(\int_0^t (\alpha_1 + \beta_1 v) \left(\int_0^t (\alpha_2 + \right. \right. \right. \right. \\
 & \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-(\alpha_2 v + \beta_2 \frac{v^2}{2})} dv - \left(\frac{\alpha_1 \alpha_2}{\alpha_3(\alpha_3 - \alpha_2)(\alpha_3 - \alpha_1)} \right) \Big) + e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \int_0^t (\alpha_2 + \\
 & \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv \left(\left(\frac{\alpha_1}{\alpha_2(\alpha_2 - \alpha_1)} \right) - \int_0^t \frac{(\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv}{\alpha_1} \right) + e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \int_0^t (\alpha_2 + \\
 & \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv \left(\left(\int_0^t (\alpha_1 + \right. \right. \\
 & \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv \int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} dv - \int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} \left(\int_0^t (\alpha_1 + \right. \\
 & \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv \right) dv \Big) + e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \left(\int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} \left(\int_0^t (\alpha_1 + \beta_1 v) \left(\int_0^t (\alpha_2 + \right. \right. \right. \right.
 \end{aligned}$$

$$\left. \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-(\alpha_2 v + \beta_2 \frac{v^2}{2})} dv) - \left(\int_0^t (\alpha_1 + \beta_1 v) \left(\int_0^t (\alpha_2 + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-(\alpha_1 v + \beta_1 \frac{v^2}{2})} dv \right) \left(\int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} dv \right) \right) \right\} \quad (32)$$

The avg. waiting time of a User in the 3rd queue is

$$W_3(t) = \frac{L_3(t)}{ThP_3(t)} \quad (33)$$

The var. of the No. of Users in the 3rd queue is

$$V_3(t) = \lambda \cdot \theta \left[e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \left(\int_0^t (\alpha_1 + \beta_1 v) \left(\int_0^t (\alpha_2 + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-(\alpha_2 v + \beta_2 \frac{v^2}{2})} dv - \left(\frac{\alpha_1 \alpha_2}{\alpha_3 (\alpha_3 - \alpha_2) (\alpha_3 - \alpha_1)} \right) \right) + e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \int_0^t (\alpha_2 + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv \left(\left(\frac{\alpha_1}{\alpha_2 (\alpha_2 - \alpha_1)} \right) - \int_0^t \frac{(\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv}{\alpha_1} \right) + e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \int_0^t (\alpha_2 + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv \left(\int_0^t (\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv \int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} dv - \int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} \left(\int_0^t (\alpha_1 + \beta_1 v) \left(\int_0^t (\alpha_2 + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-(\alpha_2 v + \beta_2 \frac{v^2}{2})} dv) - \left(\int_0^t (\alpha_1 + \beta_1 v) \left(\int_0^t (\alpha_2 + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-(\alpha_1 v + \beta_1 \frac{v^2}{2})} dv \right) \left(\int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} dv \right) \right) \right) \right] \quad (34)$$

The CV of the No. of Users in the 3rd system is

$$\begin{aligned}
 & CV_3(t) \\
 &= \left(\lambda \cdot \theta \left[e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \left(\int_0^t (\alpha_1 + \beta_1 v) \left(\int_0^t (\alpha_2 + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-(\alpha_2 v + \beta_2 \frac{v^2}{2})} dv \right. \right. \right. \right. \\
 &\quad \left. \left. \left. - \left(\frac{\alpha_1 \alpha_2}{\alpha_3 (\alpha_3 - \alpha_2) (\alpha_3 - \alpha_1)} \right) \right) \right. \right. \\
 &\quad \left. \left. + e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \int_0^t (\alpha_2 + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv \left(\left(\frac{\alpha_1}{\alpha_2 (\alpha_2 - \alpha_1)} \right) \right. \right. \right. \\
 &\quad \left. \left. \left. - \int_0^t \frac{(\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv}{\alpha_1} \right) \right. \right. \\
 &\quad \left. \left. + e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \int_0^t (\alpha_2 + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv \left(\int_0^t (\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv \right. \right. \right. \\
 &\quad \left. \left. \left. + \int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} dv \right) \right. \right. \\
 &\quad \left. \left. - \int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} \left(\int_0^t (\alpha_1 + \beta_1 v) e^{(\alpha_2 - \alpha_1)v + (\beta_2 - \beta_1)\frac{v^2}{2}} dv \right) dv \right) \right. \\
 &\quad \left. \left. + e^{-(\alpha_3 t + \beta_3 \frac{t^2}{2})} \left(\int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} \left(\int_0^t (\alpha_1 + \beta_1 v) \left(\int_0^t (\alpha_2 + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-(\alpha_2 v + \beta_2 \frac{v^2}{2})} dv \right) dv \right) \right. \right. \right. \\
 &\quad \left. \left. \left. - \left(\int_0^t (\alpha_1 + \beta_1 v) \left(\int_0^t (\alpha_2 + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-(\alpha_2 v + \beta_2 \frac{v^2}{2})} dv \right) \right. \right. \right. \right. \\
 &\quad \left. \left. \left. \left. + \beta_2 v) e^{(\alpha_3 - \alpha_2)v + (\beta_3 - \beta_2)\frac{v^2}{2}} dv e^{\alpha_2 v + \beta_2 \frac{v^2}{2}} e^{-(\alpha_1 v + \beta_1 \frac{v^2}{2})} dv \right) \left(\int_0^t e^{(\alpha_1 v + \beta_1 \frac{v^2}{2})} dv \right) \right) \right) \right)^{-1/2} \\
 &\quad * 100 \quad (35)
 \end{aligned}$$

The avg. No. of Users present in the entire queueing system at any time t is denoted by $L(t)$, and is given by the sum of the expected No. of Users in each of the three queues:

$$L(t) = L_1(t) + L_2(t) + L_3(t) \quad (36)$$

where:

- $L_1(t)$ is the avg. No. of Users in the 1st queue at time t ,
- $L_2(t)$ is the avg. No. of Users in the 2nd queue, and
- $L_3(t)$ is the avg. No. of Users in the 3rd queue.

4. NUMERICAL ILLUSTRATION AND SENSITIVITY ANALYSIS

This section presents a numerical study of the proposed three-node tandem queueing system with probabilistic routing. In this model, customers arrive at the first queue according to a Poisson process and receive service at the first service station. After completion of service at the first node, the customer either joins the second queue with probability θ or leaves the system with probability $(1-\theta)$. Customers entering the second queue subsequently proceed to the third queue after service. The service mechanisms at all three nodes are governed by non-homogeneous Poisson processes with time-dependent service rates given by $\mu_1(t) = \alpha_1 + \beta_1(t)$, $\mu_2(t) = \alpha_2 + \beta_2(t)$ and $\mu_3(t) = \alpha_3 + \beta_3(t)$ respectively.

The computed results for these performance indicators, corresponding to different parameter values, are summarized in Table 1. The relationships between the parameters and the observed performance metrics are visually depicted in Figure 2.

Effect of Time (t)

As time t increases from **0.100 to 0.104**, the probability of emptiness of the system increases from **2.68219 to 2.76264**. The probability that the first queue is empty decreases from **0.93209 to 0.93044**, while the probability that the second queue is empty increases from **0.99964 to 0.99978** and the probability that the third queue is empty increases from **0.75046 to 0.83242**. The mean number of customers in the first queue increases from **0.07032 to 0.07210**, whereas in the second queue it decreases from **0.00036 to 0.00022**, and in the third queue it decreases from **0.28707 to 0.18342**. The total system size decreases from **0.35775 to 0.25574**.

Effect of Arrival Rate (λ)

As the arrival rate λ increases from **0.30 to 1.00**, the probability of emptiness of the first queue decreases from **0.98951 to 0.96545**, the probability of emptiness of the second queue decreases from **0.99995 to 0.99982**, and the probability of emptiness of the third queue decreases from **0.95785 to 0.86629**.

The mean number of customers in the first queue increases from **0.01055 to 0.03516**, in the second queue from **0.00005 to 0.00018**, and in the third queue from **0.04306 to 0.14354**, resulting in an increase in total system size from **0.05366 to 0.17888**.

Effect of Routing Probability (θ)

As the routing probability θ increases from **0.10 to 0.90**, the probability of emptiness of the third queue decreases significantly from **0.75046 to 0.05666**, while the probabilities of emptiness of the first and second queues show slight variations.

The mean number of customers in the first queue remains nearly constant at **0.07032**, whereas in the second queue it increases from **0.00036 to 0.00359**, and in the third queue it increases sharply from **0.28707 to 2.87071**. The total system size increases from **0.35775 to 2.94462**.

Effect of Service Rate Parameter (α_1)

As α_1 increases from **20.0 to 20.4**, the probability of emptiness of the third queue decreases from **0.05666 to 0.00067**, while the probabilities of emptiness of the first and second queues show marginal changes.

The mean number of customers in the third queue increases significantly from **2.87071 to 7.30818**, while the first and second queues show only slight variations. The total system size increases from **2.94462 to 7.38427**.

Effect of Service Rate Parameter (β_1)

As β_1 decreases from **17 to 10**, the probability of emptiness of the first queue decreases slightly from **0.93196 to 0.93103**, while the other queues show minimal changes.

The mean number of customers in the third queue increases from **2.59299 to 2.65638**, indicating mild congestion growth in downstream nodes.

Effect of Service Rate Parameter (α_2)

As α_2 decreases from **25 to 23**, the probability of emptiness of the second queue decreases from **0.99667 to 0.97260**, and the probability of emptiness of the third queue decreases from **0.07020 to 0.00042**.

The mean number of customers in the third queue increases significantly from **2.65638 to 7.78311**, leading to an increase in total system size from **2.73118 to 7.88236**.

Effect of Service Rate Parameter (β_2)

As β_2 increases from **18 to 45**, the probability of emptiness of the second queue increases from **0.99667 to 0.99795**, while the probability of emptiness of the third queue decreases slightly.

The mean number of customers in the second queue decreases from **0.00333 to 0.00205**, while the third queue increases from **2.68093 to 2.82317**.

Effect of Service Rate Parameter (α_3)

As α_3 increases from **24 to 24.7**, the probability of emptiness of the third queue decreases from **0.74828 to 0.00017**, while the first and second queues remain nearly unchanged.

The mean number of customers in the third queue increases sharply from **0.28998 to 8.70858**, leading to a significant rise in total system size.

Effect of Service Rate Parameter (β_3)

As β_3 increases from **14 to 35 and 70**, the probability of emptiness of the third queue increases gradually, while the mean number of customers in the third queue decreases from **5.30192 to 3.02609**.

This results in a reduction in total system size from **5.39041 to 3.11135**.

The above numerical results clearly indicate that the routing probability θ plays a significant role in controlling the flow of customers in the system, and higher values of θ lead to increased congestion in downstream queues, particularly in the third node

Table:1 Values of $P_{000}(t)$, $P_{0..}(t)$, $P_{.0}(t)$, $P_{..0}(t)$, $L_1(t)$, $L_2(t)$, $L_3(t)$ and $L(t)$ for various values of parameters

t	λ	θ	α_1	β_1	α_2	β_2	α_3	β_3	$P_{000}(t)$	$P_{0..}(t)$	$P_{.0}(t)$	$P_{..0}(t)$	$L_1(t)$	$L_2(t)$	$L_3(t)$	$L(t)$
0.100	2	0.1	20	18	25	18	21	19	2.68219	0.93209	0.99964	0.75046	0.07032	0.00036	0.28707	0.35775
0.101	2	0.1	20	18	25	18	21	19	2.70019	0.93166	0.99968	0.76885	0.07079	0.00032	0.26286	0.33397
0.102	2	0.1	20	18	25	18	21	19	2.71952	0.93124	0.99971	0.78857	0.07124	0.00029	0.23754	0.30907
0.103	2	0.1	20	18	25	18	21	19	2.74029	0.93083	0.99974	0.80972	0.07167	0.00026	0.21107	0.28300
0.104	2	0.1	20	18	25	18	21	19	2.76264	0.93044	0.99978	0.83242	0.0721	0.00022	0.18342	0.25574
0.100	0.300	0.1	20	18	25	18	21	19	2.94731	0.98951	0.99995	0.95785	0.01055	0.00005	0.04306	0.05366
0.100	0.500	0.1	20	18	25	18	21	19	2.91323	0.98257	0.99991	0.93075	0.01758	0.00009	0.07177	0.08944
0.100	0.700	0.1	20	18	25	18	21	19	2.87997	0.97569	0.99987	0.90441	0.02461	0.00013	0.10047	0.12521
0.100	0.900	0.1	20	18	25	18	21	19	2.8475	0.96885	0.99984	0.87881	0.03165	0.00016	0.12918	0.16099
0.100	1.000	0.1	20	18	25	18	21	19	2.83156	0.96545	0.99982	0.86629	0.03516	0.00018	0.14354	0.17888
0.100	2	0.200	20	18	25	18	21	19	2.49456	0.93209	0.99928	0.56319	0.07032	0.00072	0.57414	0.64518
0.100	2	0.400	20	18	25	18	21	19	2.24783	0.93209	0.99856	0.31718	0.07032	0.00144	1.14828	1.22004
0.100	2	0.600	20	18	25	18	21	19	2.10857	0.93209	0.99785	0.17863	0.07032	0.00215	1.72243	1.79490
0.100	2	0.800	20	18	25	18	21	19	2.02982	0.93209	0.99713	0.1006	0.07032	0.00287	2.29657	2.36976
0.100	2	0.900	20	18	25	18	21	19	1.98517	0.93209	0.99642	0.05666	0.07032	0.00359	2.87071	2.94462

t	λ	θ	α_1	β_1	α_2	β_2	α_3	β_3	P00 0(t)	P0.. (t)	P.0. (t)	P..0 (t)	L1(t)	L2(t)	L3(t)	L(t)
0. 10 0	2	0. 9 0	2 0. 0	1 7	2 5	1 8	2 1	1 9	2.0 035 2	0.9 319 6	0.9 967 6	0.0 748	0.0 704 7	0.0 032 4	2.5 929 9	2.6 667 0
0. 10 0	2	0. 9 0	2 0. 1	1 7	2 5	1 8	2 1	1 9	1.9 625 1	0.9 320 5	0.9 961 1	0.0 343 5	0.0 703 7	0.0 039	3.3 712 4	3.4 455 1
0. 10 0	2	0. 9 0	2 0. 2	1 7	2 5	1 8	2 1	1 9	1.9 404 8	0.9 321 4	0.9 954 3	0.0 129 1	0.0 702 8	0.0 045 8	4.3 497 5	4.4 246 1
0. 10 0	2	0. 9 0	2 0. 3	1 7	2 5	1 8	2 1	1 9	1.9 306 2	0.9 322 3	0.9 947 4	0.0 036 5	0.0 701 8	0.0 052 8	5.6 143 7	5.6 898 3
0. 10 0	2	0. 9 0	2 0. 4	1 7	2 5	1 8	2 1	1 9	1.9 27	0.9 323 2	0.9 940 1	0.0 006 7	0.0 700 8	0.0 060 1	7.3 081 8	7.3 842 7
0. 10 0	2	0. 9 0	2 0. 0	1 6. 0	2 5	1 8	2 1	1 9	2.0 026 9	0.9 318 3	0.9 967 5	0.0 741 1	0.0 706 1	0.0 032 6	2.6 022 6	2.6 761 3
0. 10 0	2	0. 9 0	2 0. 0	1 5. 0	2 5	1 8	2 1	1 9	2.0 018 6	0.9 316 9	0.9 967 4	0.0 734 3	0.0 707 5	0.0 032 7	2.6 114 7	2.6 854 9
0. 10 0	2	0. 9 0	2 0. 0	1 4. 0	2 5	1 8	2 1	1 9	2.0 010 4	0.9 315 6	0.9 967 2	0.0 727 6	0.0 708 9	0.0 032 8	2.6 206	2.6 947 7
0. 10 0	2	0. 9 0	2 0. 0	1 2. 0	2 5	1 8	2 1	1 9	1.9 994 5	0.9 312 9	0.9 967	0.0 714 6	0.0 711 8	0.0 033 1	2.6 386 4	2.7 131 3
0. 10 0	2	0. 9 0	2 0. 0	1 1. 0	2 5	1 8	2 1	1 9	1.9 986 6	0.9 311 6	0.9 966 8	0.0 708 2	0.0 713 2	0.0 033 2	2.6 475 5	2.7 221 9
0. 10 0	2	0. 9 0	2 0. 0	1 0	2 5. 0	1 8	2 1	1 9	1.9 979	0.9 310 3	0.9 966 7	0.0 702	0.0 714 7	0.0 033 3	2.6 563 8	2.7 311 8
0. 10 0	2	0. 9 0	2 0. 0	1 0	2 4. 5	1 8	2 1	1 9	1.9 566 8	0.9 310 3	0.9 929 6	0.0 326 9	0.0 714 7	0.0 070 6	3.4 208	3.4 993 3
0. 10 0	2	0. 9 0	2 0. 0	1 0	2 4. 0	1 8	2 1	1 9	1.9 312 3	0.9 310 3	0.9 881	0.0 121	0.0 714 7	0.0 119 7	4.4 147	4.4 981 4
0. 10 0	2	0. 9 0	2 0. 0	1 0	2 3. 5	1 8	2 1	1 9	1.9 157 2	0.9 310 3	0.9 815 9	0.0 031	0.0 714 7	0.0 185 8	5.7 765 9	5.8 666 4
0. 10 0	2	0. 9 0	2 0. 0	1 0	2 3. 0	1 8	2 1	1 9	1.9 040 5	0.9 310 3	0.9 726	0.0 004 2	0.0 714 7	0.0 277 8	7.7 831 1	7.8 823 6
0. 10 0	2	0. 9 0	2 0. 0	1 0	2 5	2 2	2 1	1 9	1.9 964	0.9 310 3	0.9 968 7	0.0 685	0.0 714 7	0.0 031 3	2.6 809 3	2.7 555 3
0. 10 0	2	0. 9 0	2 0. 0	1 0	2 5	2 3	2 1	1 9	1.9 960 3	0.9 310 3	0.9 969 2	0.0 680 8	0.0 714 7	0.0 030 8	2.6 870 7	2.7 616 2

t	λ	θ	α_1	β_1	α_2	β_2	α_3	β_3	P00 0(t)	P0.. (t)	P.0. (t)	P..0 (t)	L1(t)	L2(t)	L3(t)	L(t)
0. 10 0	2	0. 9 0	2 0	1 0	2 5	2 4	2 1	1 9	1.9 956 6	0.9 310 3	0.9 969 7	0.0 676 6	0.0 714 7	0.0 030 3	2.6 932 2	2.7 677 2
0. 10 0	2	0. 9 0	2 0	1 0	2 5	3 5	2 1	1 9	1.9 917 5	0.9 310 3	0.9 975	0.0 632 2	0.0 714 7	0.0 025	2.7 610 6	2.8 350 3
0. 10 0	2	0. 9 0	2 0	1 0	2 5	4 5	2 1	1 9	1.9 884	0.9 310 3	0.9 979 5	0.0 594 2	0.0 714 7	0.0 020 5	2.8 231 7	2.8 966 9
0. 10 0	2	0. 9 0	2 0	1 0	2 5	1 8	2 4	1 9	2.6 759 8	0.9 310 3	0.9 966 7	0.7 482 8	0.0 714 7	0.0 033 3	0.2 899 8	0.3 647 8
0. 10 0	2	0. 9 0	2 0	1 0	2 5	1 8	2 4.	1 9	2.2 553 1	0.9 310 3	0.9 966 7	0.3 276 1	0.0 714 7	0.0 033 3	1.1 159 4	1.1 907 4
0. 10 0	2	0. 9 0	2 0	1 0	2 5	1 8	2 4.	1 9	2.0 037 1	0.9 310 3	0.9 966 7	0.0 760 1	0.0 714 7	0.0 033 3	2.5 769 2	2.6 517 2
0. 10 0	2	0. 9 0	2 0	1 0	2 5	1 8	2 4.	1 9	1.9 313 4	0.9 310 3	0.9 966 7	0.0 036 4	0.0 714 7	0.0 033 3	5.6 153 9	5.6 901 9
0. 10 0	2	0. 9 0	2 0	1 0	2 5	1 8	2 4.	1 9	1.9 278 7	0.9 310 3	0.9 966 7	0.0 001 7	0.0 714 7	0.0 033 3	8.7 085 8	8.7 833 8
0. 10 0	2	0. 9 0	2 3	1 6	2 6	1 8	2 5	1 4	1.9 189 3	0.9 349 3	0.9 790 2	0.0 049 8	0.0 672 8	0.0 212 1	5.3 019 2	5.3 904 1
0. 10 0	2	0. 9 0	2 3	1 6	2 6	1 8	2 5	1 5	1.9 191 9	0.9 349 3	0.9 790 2	0.0 052 4	0.0 672 8	0.0 212 1	5.2 513 7	5.3 398 6
0. 10 0	2	0. 9 0	2 3	1 6	2 6	1 8	2 5	1 6	1.9 194 6	0.9 349 3	0.9 790 2	0.0 055 1	0.0 672 8	0.0 212 1	5.2 009 5	5.2 894 4
0. 10 0	2	0. 9 0	2 3	1 6	2 6	1 8	2 5	1 7	1.9 197 5	0.9 349 3	0.9 790 2	0.0 058	0.0 672 8	0.0 212 1	5.1 506 8	5.2 391 7
0. 10 0	2	0. 9 0	2 3	1 6	2 6	1 8	2 5	1 8	1.9 200 4	0.9 349 3	0.9 790 2	0.0 060 9	0.0 672 8	0.0 212 1	5.1 005 4	5.1 890 3
0. 10 0	2	0. 9 0	2 3	1 6	2 6	1 8	2 5	2 2	1.9 213 9	0.9 349 3	0.9 790 2	0.0 074 4	0.0 672 8	0.0 212 1	4.9 014 1	4.9 899 0
0. 10 0	2	0. 9 0	2 4	1 6	2 7	1 9	2 5	3 5	1.9 251 6	0.9 361 1	0.9 809 4	0.0 081 1	0.0 660 2	0.0 192 4	4.8 142 9	4.8 995 5
0. 10 0	2	0. 9 0	2 5	1 7	2 8	2 0	2 6	3 5	1.9 423 9	0.9 374 5	0.9 827	0.0 222 4	0.0 646	0.0 174 5	3.8 060 8	3.8 881 3
0. 10 0	2	0. 9 0	2 4	1 6	2 7	1 9	2 5	7 0	1.9 655 6	0.9 361 1	0.9 809 4	0.0 485 1	0.0 660 2	0.0 192 4	3.0 260 9	3.1 113 5

t	λ	θ	α_1	β_1	α_2	β_2	α_3	β_3	P00 0(t)	P0.. (t)	P.0. (t)	P..0 (t)	L1(t)	L2(t)	L3(t)	L(t)
0.100	2	0.90	2.4	1.6	2.7	1.9	2.5	3.5	1.92516	0.93611	0.98094	0.00811	0.06602	0.01924	4.81429	4.89955

According to the data presented in **Table 2**, presents the numerical results of utilization rates, throughput, and average waiting times for the three-node tandem queueing system with probabilistic routing. The performance measures $U_1(t)$, $U_2(t)$, $U_3(t)$, $Thp_1(t)$, $Thp_2(t)$, $Thp_3(t)$, $W_1(t)$, $W_2(t)$ and $W_3(t)$ are computed for different values of parameters t , λ , α_i , β_i and θ

Effect of Time (t)

As time t increases from **0.100 to 0.104**, the utilization of the first queue increases from **0.06791 to 0.06956**, while the utilization of the second queue decreases from **0.00036 to 0.00022**, and the utilization of the third queue decreases from **0.24954 to 0.16758**.

The throughput of the first node increases from **1.48041 to 1.52152**, whereas the throughput of the second node decreases from **0.00962 to 0.00603**, and the throughput of the third node decreases from **5.71451 to 3.85034**.

The average waiting time in the first and second queues shows negligible variation, while in the third queue it decreases from **0.01329 to 0.00950**.

Effect of Arrival Rate (λ)

As the arrival rate λ increases from **0.30 to 1.00**, the utilization of all three queues increases, with $U_1(t)$ rising from **0.01049 to 0.03455**, $U_2(t)$ from **0.00005 to 0.00018**, and $U_3(t)$ from **0.04215 to 0.13371**.

Similarly, the throughput of all nodes increases, particularly at the third node where it rises from **0.96516 to 3.06196**.

The waiting time in the first and third queues increases slightly, indicating growing congestion as arrival intensity increases.

Effect of Routing Probability (θ)

As the routing probability θ increases from **0.10 to 0.90**, the utilization of the first node remains constant at **0.06791**, while the utilization of the second node increases from **0.00036 to 0.00358**, and the utilization of the third node increases significantly from **0.24954 to 0.94334**.

The throughput of the second node increases from **0.00962 to 0.09606**, and the throughput of the third node increases sharply from **5.71451 to 21.60252**.

The waiting time in the third queue increases from **0.01329 to 0.13289**, while the waiting times in the first and second queues remain nearly unchanged.

Effect of Service Rate Parameter (α_1)

As α_1 increases from **20.0 to 20.4**, the utilization of the third node increases from **0.94334 to 0.99933**, while the first node shows a slight decrease.

The throughput of the third node increases from **21.60252 to 22.88466**, and the waiting time in the third queue increases from **0.13289 to 0.31935**, indicating accumulation of customers in the final stage.

Effect of Service Rate Parameter (β_1)

As β_1 decreases from **17 to 10**, the utilization of the first node increases slightly from **0.06804 to 0.06897**, while other nodes show marginal changes.

The throughput of the first node decreases slightly, while the third node throughput remains nearly stable, indicating limited sensitivity to this parameter.

Effect of Service Rate Parameter (α_2)

As α_2 decreases from **25 to 23**, the utilization of the second node increases from **0.00333 to 0.02740**, and the utilization of the third node approaches unity.

The throughput of the second node increases significantly from **0.08922 to 0.67940**, and the waiting time in the third queue increases from **0.12476 to 0.34002**.

Effect of Service Rate Parameter (β_2)

As β_2 increases from **18 to 45**, the utilization of the second node decreases from **0.00333 to 0.00205**, while the utilization of the third node increases from **0.93150 to 0.94058**.

The throughput of the second node decreases, while the third node throughput increases slightly, indicating redistribution of load.

Effect of Service Rate Parameter (α_3)

As α_3 increases from **24 to 24.7**, the utilization of the third node increases sharply from **0.25172 to 0.99983**, and the throughput of the third node increases from **6.51954 to 26.59561**.

The waiting time in the third queue increases from **0.04448 to 0.32744**, indicating heavy congestion at higher service intensities.

Effect of Service Rate Parameter (β_3)

As β_3 increases from **14 to 70**, the utilization of the third node decreases from **0.99502 to 0.95149**, while the throughput increases from **26.26848 to 30.44784**.

The waiting time in the third queue decreases from **0.20184 to 0.09939**, indicating improved service efficiency at higher growth rates.

The above results clearly show that the routing probability θ has a significant impact on system performance, particularly on utilization, throughput, and waiting time of downstream nodes, where higher values of θ lead to increased congestion and processing load at the third service station.

Table:2 Values of $U_1(t)$, $U_2(t)$, $U_3(t)$, $Thp_1(t)$, $Thp_2(t)$, $Thp_3(t)$, $W_1(t)$, $W_2(t)$ and $W_3(t)$ for different values of parameters

t	λ	θ	α_1	β_1	α_2	β_2	α_3	β_3	$U_1(t)$	$U_2(t)$	$U_3(t)$	$Thp_1(t)$	$Thp_2(t)$	$Thp_3(t)$	$W_1(t)$	$W_2(t)$	$W_3(t)$
0.100	2	0.1	20	18	25	18	21	19	0.06791	0.0036	0.24954	1.48041	0.00962	5.71451	0.0475	0.03732	0.01329
0.101	2	0.1	20	18	25	18	21	19	0.06834	0.0032	0.23115	1.49101	0.00867	5.29777	0.04748	0.03729	0.01236
0.102	2	0.1	20	18	25	18	21	19	0.06876	0.0029	0.21143	1.50139	0.00775	4.84985	0.04745	0.03727	0.01142
0.103	2	0.1	20	18	25	18	21	19	0.06917	0.0026	0.19028	1.51156	0.00687	4.36828	0.04742	0.03724	0.01046
0.104	2	0.1	20	18	25	18	21	19	0.06956	0.0022	0.16758	1.52152	0.00603	3.85034	0.04739	0.03722	0.0095
0.100	0.30	0.1	20	18	25	18	21	19	0.01049	0.0005	0.04215	0.2875	0.00144	0.96516	0.04611	0.03731	0.00537
0.100	0.50	0.1	20	18	25	18	21	19	0.01743	0.0009	0.06925	0.3792	0.00241	1.58589	0.04628	0.03732	0.00612
0.100	0.70	0.1	20	18	25	18	21	19	0.02431	0.0013	0.09559	0.5302	0.0037	2.18906	0.04644	0.03732	0.00692

t	λ	θ	α_1	β_1	α_2	β_2	α_3	β_3	$U_1(t)$	$U_2(t)$	$U_3(t)$	$Th_{p1}(t)$	$Th_{p2}(t)$	$Th_{p3}(t)$	$W_1(t)$	$W_2(t)$	$W_3(t)$
0.1000	0.90	0.1	20	18	25	18	21	19	0.03115	0.0016	0.12119	0.67908	0.00433	2.77516	0.0466	0.03732	0.00778
0.1000	1.00	0.1	20	18	25	18	21	19	0.03455	0.0018	0.13371	0.75321	0.00481	3.06196	0.04668	0.03732	0.00823
0.1000	2	0.20	20	18	25	18	21	19	0.06791	0.0072	0.43681	1.48041	0.01924	10.003	0.0475	0.03733	0.02658
0.1000	2	0.40	20	18	25	18	21	19	0.06791	0.0044	0.68282	1.48041	0.03847	15.63657	0.0475	0.03734	0.05316
0.1000	2	0.60	20	18	25	18	21	19	0.06791	0.0015	0.82137	1.48041	0.05768	18.80933	0.0475	0.03735	0.07973
0.1000	2	0.80	20	18	25	18	21	19	0.06791	0.0087	0.8994	1.48041	0.07688	20.59618	0.0475	0.03737	0.10631
0.1000	2	0.90	20	18	25	18	21	19	0.06791	0.0058	0.94334	1.48041	0.09606	21.60252	0.0475	0.03738	0.13289
0.1000	2	0.900	20	17	25	18	21	19	0.06804	0.0024	0.9252	1.47648	0.0868	21.18717	0.04773	0.03737	0.12238
0.1000	2	0.901	20	17	25	18	21	19	0.06795	0.0089	0.96565	1.48138	0.10431	22.11346	0.0475	0.03739	0.15245
0.1000	2	0.902	20	17	25	18	21	19	0.06786	0.0057	0.98709	1.48623	0.12239	22.60436	0.04729	0.0374	0.19243
0.1000	2	0.903	20	17	25	18	21	19	0.06777	0.0026	0.99635	1.49104	0.1411	22.81653	0.04707	0.03741	0.24607
0.1000	2	0.904	20	17	25	18	21	19	0.06768	0.0099	0.99933	1.4958	0.16046	22.88466	0.04685	0.03743	0.31935
0.1000	2	0.90	20	16.0	25	18	21	19	0.06817	0.0025	0.92589	1.47253	0.08713	21.20298	0.04795	0.03737	0.12273

t	λ	θ	α_1	β_1	α_2	β_2	α_3	β_3	$U_1(t)$	$U_2(t)$	$U_3(t)$	$Th_{p1}(t)$	$Th_{p2}(t)$	$Th_{p3}(t)$	$W_1(t)$	$W_2(t)$	$W_3(t)$
0.1000	2	0.90	20	15.0	25	18	21	19	0.06831	0.00326	0.92657	1.46857	0.08747	21.21853	0.04818	0.03737	0.12307
0.1000	2	0.90	20	14.0	25	18	21	19	0.06844	0.00328	0.92724	1.46458	0.08781	21.23381	0.0484	0.03737	0.12342
0.1000	2	0.90	20	12.0	25	18	21	19	0.06871	0.0033	0.92854	1.45656	0.08851	21.2636	0.04887	0.03738	0.12409
0.1000	2	0.90	20	11.0	25	18	21	19	0.06884	0.00332	0.92918	1.45251	0.08886	21.27811	0.0491	0.03738	0.12443
0.1000	2	0.90	20	10	25.0	18	21	19	0.06897	0.00333	0.9298	1.44845	0.08922	21.29238	0.04934	0.03738	0.12476
0.1000	2	0.90	20	10	24.5	18	21	19	0.06897	0.00704	0.96731	1.44845	0.18509	22.15149	0.04934	0.03816	0.15443
0.1000	2	0.90	20	10	24.0	18	21	19	0.06897	0.0119	0.9879	1.44845	0.30696	22.62295	0.04934	0.03899	0.19514
0.1000	2	0.90	20	10	23.5	18	21	19	0.06897	0.01841	0.9969	1.44845	0.46576	22.82903	0.04934	0.03989	0.25304
0.1000	2	0.90	20	10	23.0	18	21	19	0.06897	0.0274	0.99958	1.44845	0.6794	22.89046	0.04934	0.04089	0.34002
0.1000	2	0.90	20	10	25	22	21	19	0.06897	0.00313	0.9315	1.44845	0.08501	21.33136	0.04934	0.03682	0.12568
0.1000	2	0.90	20	10	25	23	21	19	0.06897	0.00308	0.93192	1.44845	0.08396	21.34097	0.04934	0.03669	0.12591
0.1000	2	0.90	20	10	25	24	21	19	0.06897	0.00303	0.93234	1.44845	0.0829	21.35052	0.04934	0.03655	0.12614
0.1000	2	0.90	20	10	25	35	21	19	0.06897	0.0025	0.93678	1.44845	0.07119	21.45216	0.04934	0.03513	0.12871

t	λ	θ	α_1	β_1	α_2	β_2	α_3	β_3	$U_1(t)$	$U_2(t)$	$U_3(t)$	$Th_{p1}(t)$	$Th_{p2}(t)$	$Th_{p3}(t)$	$W_1(t)$	$W_2(t)$	$W_3(t)$
0.1000	2	0.90	20	10	25	45	21	19	0.06897	0.00205	0.94058	1.44845	0.06044	21.53935	0.04934	0.03393	0.13107
0.1000	2	0.90	20	10	25	18	24	19	0.06897	0.00333	0.25172	1.44845	0.08922	6.51954	0.04934	0.03738	0.04448
0.1000	2	0.90	20	10	25	18	24.2	19	0.06897	0.00333	0.67239	1.44845	0.08922	17.54943	0.04934	0.03738	0.06359
0.1000	2	0.90	20	10	25	18	24.4	19	0.06897	0.00333	0.92399	1.44845	0.08922	24.301	0.04934	0.03738	0.10604
0.1000	2	0.90	20	10	25	18	24.6	19	0.06897	0.00333	0.99636	1.44845	0.08922	26.4035	0.04934	0.03738	0.21268
0.1000	2	0.90	20	10	25	18	24.7	19	0.06897	0.00333	0.99983	1.44845	0.08922	26.59561	0.04934	0.03738	0.32744
0.1000	2	0.90	23	16	26	18	25	14	0.06507	0.02098	0.99502	1.60072	0.58331	26.26848	0.04203	0.03635	0.20184
0.1000	2	0.90	23	16	26	18	25	15	0.06507	0.02098	0.99476	1.60072	0.58331	26.36113	0.04203	0.03635	0.19921
0.1000	2	0.90	23	16	26	18	25	16	0.06507	0.02098	0.9949	1.60072	0.58331	26.4534	0.04203	0.03635	0.19661
0.1000	2	0.90	23	16	26	18	25	17	0.06507	0.02098	0.9942	1.60072	0.58331	26.54526	0.04203	0.03635	0.19403
0.1000	2	0.90	23	16	26	18	25	18	0.06507	0.02098	0.99391	1.60072	0.58331	26.6367	0.04203	0.03635	0.19149
0.1000	2	0.90	23	16	26	18	25	22	0.06507	0.02098	0.99256	1.60072	0.58331	26.99774	0.04203	0.03635	0.18155
0.1000	2	0.90	24	16	27	19	25	35	0.06389	0.01906	0.99189	1.6356	0.55083	28.26878	0.04037	0.03494	0.1703

t	λ	θ	α_1	β_1	α_2	β_2	α_3	β_3	$U_1(t)$	$U_2(t)$	$U_3(t)$	$Th_{p1}(t)$	$Th_{p2}(t)$	$Th_{p3}(t)$	$W_1(t)$	$W_2(t)$	$W_3(t)$
0.1000	2	0.90	25	17	28	20	26	35	0.06255	0.0173	0.9776	1.67018	0.51892	28.84406	0.03868	0.03362	0.13195
0.1000	2	0.90	24	16	27	19	25	70	0.06389	0.01906	0.95149	1.6356	0.55083	30.44784	0.04037	0.03494	0.09939
0.1000	2	0.90	24	16	27	19	25	35	0.06389	0.01906	0.99189	1.6356	0.55083	28.26878	0.04037	0.03494	0.1703

Table 3 presents the numerical results of variance and coefficient of variation of the number of customers in the three-node tandem queueing system. The performance measures $V_1(t)$, $V_2(t)$, $V_3(t)$, $CV_1(t)$, $CV_2(t)$ and $CV_3(t)$ are evaluated for different values of system parameters.

Effect of Time (t)

As time t increases from **0.100 to 0.104**, the variance in the first queue increases from **0.07032 to 0.07210**, while the variance in the second queue decreases from **0.00036 to 0.00022**, and the variance in the third queue decreases from **0.28707 to 0.18342**.

The coefficient of variation in the first queue decreases from **3.77092 to 3.72412**, whereas in the second queue it increases from **52.77160 to 66.73744**, and in the third queue it increases from **1.86640 to 2.33495**.

Effect of Arrival Rate (λ)

As the arrival rate λ increases from **0.30 to 1.00**, the variance in the first queue increases from **0.01055 to 0.03516**, in the second queue from **0.00005 to 0.00018**, and in the third queue from **0.04306 to 0.14354**.

The coefficient of variation in all queues decreases, with $CV_1(t)$ decreasing from **9.73647 to 5.33289**, $CV_2(t)$ from **136.25568 to 74.63031**, and $CV_3(t)$ from **4.81903 to 2.63949**.

Effect of Routing Probability (θ)

As the routing probability θ increases from **0.10 to 0.90**, the variance in the first queue remains constant at **0.07032**, while the variance in the second queue increases from **0.00036 to 0.00359**, and in the third queue increases significantly from **0.28707 to 2.87071**.

The coefficient of variation in the second queue decreases from **52.77160 to 16.68784**, and in the third queue decreases from **1.86640 to 0.59021**, while the first queue remains unchanged.

Effect of Service Rate Parameter (α_1)

As α_1 increases from **20.0 to 20.4**, the variance in the third queue increases from **2.59299 to 7.30818**, while the first queue shows a slight decrease.

The coefficient of variation in the third queue decreases from **0.62101 to 0.36991**, indicating reduced relative variability despite higher absolute variance.

Effect of Service Rate Parameter (β_1)

As β_1 decreases from **17 to 10**, the variance in the first queue increases slightly from **0.07047 to 0.07147**, while the second and third queues show minor increases.

The coefficient of variation in all queues decreases slightly, indicating marginal stabilization.

Effect of Service Rate Parameter (α_2)

As α_2 decreases from **25 to 23**, the variance in the second queue increases from **0.00333 to 0.02778**, and in the third queue increases from **2.65638 to 7.78311**.

The coefficient of variation in the second queue decreases from **17.31738 to 6.00003**, and in the third queue decreases from **0.61356 to 0.35845**.

Effect of Service Rate Parameter (β_2)

As β_2 increases from **18 to 45**, the variance in the second queue decreases from **0.00333 to 0.00205**, while the variance in the third queue increases from **2.68093 to 2.82317**.

The coefficient of variation in the second queue increases from **17.31738 to 22.08168**, while in the third queue it decreases slightly.

Effect of Service Rate Parameter (α_3)

As α_3 increases from **24 to 24.7**, the variance in the third queue increases significantly from **0.28998 to 8.70858**.

The coefficient of variation in the third queue decreases from **1.85702 to 0.33886**, indicating reduced dispersion relative to mean.

Effect of Service Rate Parameter (β_3)

As β_3 increases from **14 to 70**, the variance in the third queue decreases from **5.30192 to 3.02609**, while the coefficient of variation increases from **0.43429 to 0.57486**.

The above results indicate that the routing probability θ significantly influences the variability of the system, where higher values of θ increase the variance in downstream queues while reducing the relative variability as measured by the coefficient of variation.

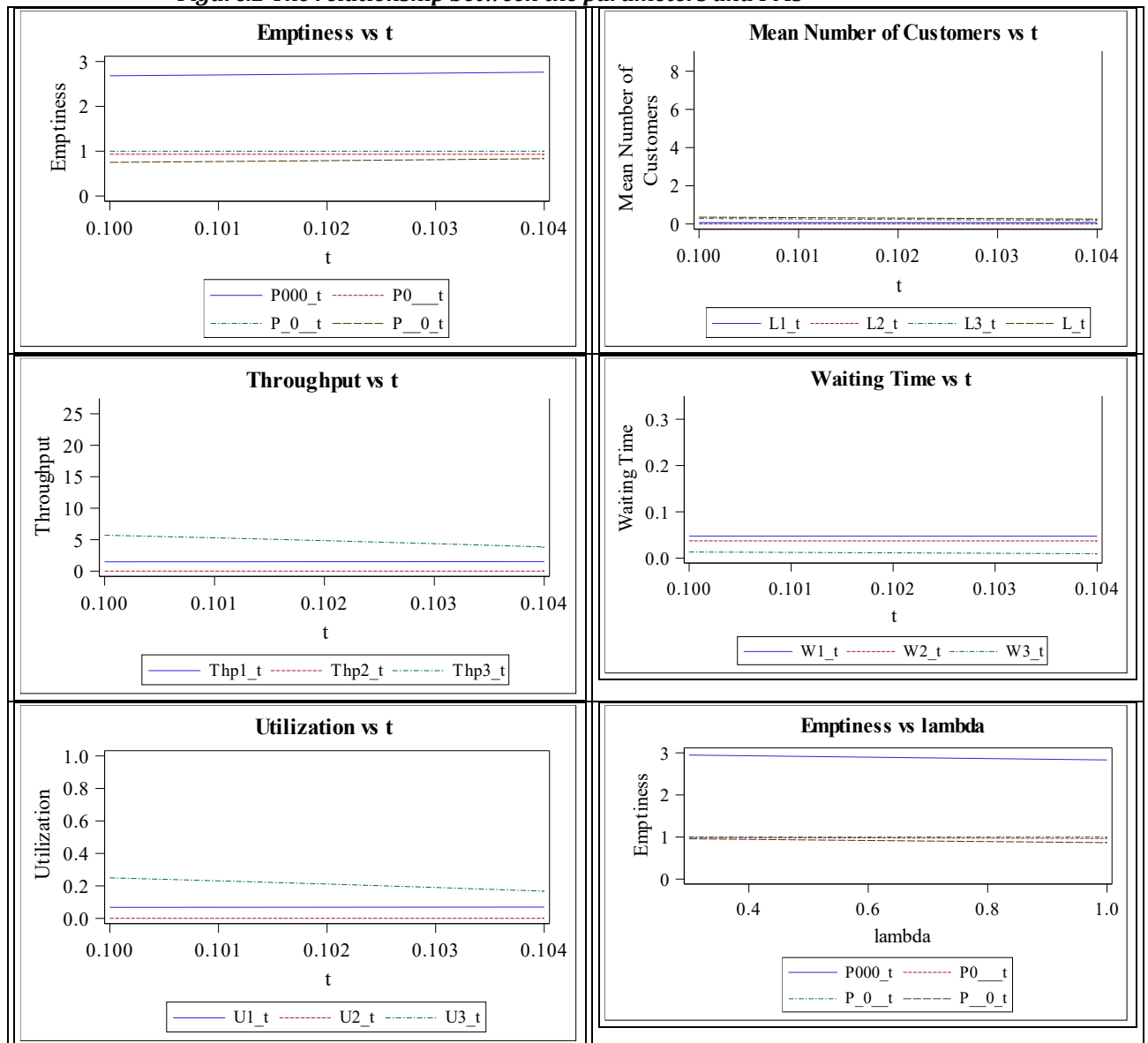
Table:3 Values of $V_1(t)$, $V_2(t)$, $V_3(t)$, $CV_1(t)$, $CV_2(t)$ and $CV_3(t)$, for different values of parameters

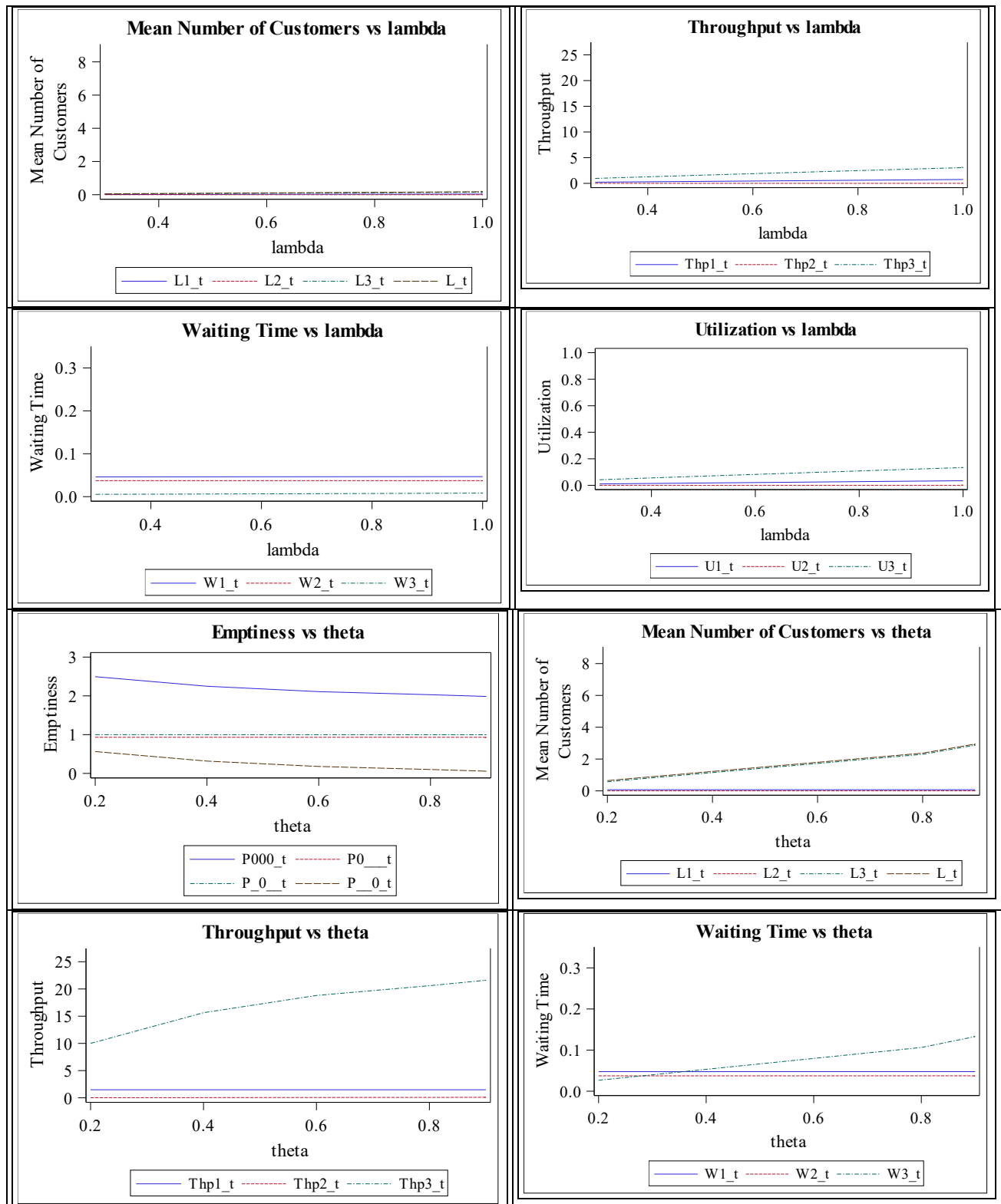
t	λ	θ	α_1	β_1	α_2	β_2	α_3	β_3	$V_1(t)$	$V_2(t)$	$V_3(t)$	$CV_1(t)$	$CV_2(t)$	$CV_3(t)$
0.100	2	0.1	20	18	25	18	21	19	0.07032	0.00036	0.28707	3.77092	52.77160	1.86640
0.101	2	0.1	20	18	25	18	21	19	0.07079	0.00032	0.26286	3.75861	55.61903	1.95046
0.102	2	0.1	20	18	25	18	21	19	0.07124	0.00029	0.23754	3.74672	58.83126	2.05179
0.103	2	0.1	20	18	25	18	21	19	0.07167	0.00026	0.21107	3.73523	62.49666	2.17665
0.104	2	0.1	20	18	25	18	21	19	0.07210	0.00022	0.18342	3.72412	66.73744	2.33495
0.100	0.30	0.1	20	18	25	18	21	19	0.01055	0.00005	0.04306	9.73647	136.25568	4.81903
0.100	0.50	0.1	20	18	25	18	21	19	0.01758	0.00009	0.07177	7.54184	105.54320	3.73280
0.100	0.70	0.1	20	18	25	18	21	19	0.02461	0.00013	0.10047	6.37402	89.20028	3.15479
0.100	0.90	0.1	20	18	25	18	21	19	0.03165	0.00016	0.12918	5.62136	78.66725	2.78227
0.100	1.00	0.1	20	18	25	18	21	19	0.03516	0.00018	0.14354	5.33289	74.63031	2.63949
0.100	2	0.20	20	18	25	18	21	19	0.07032	0.00072	0.57414	3.77092	37.31515	1.31975
0.100	2	0.40	20	18	25	18	21	19	0.07032	0.00144	1.14828	3.77092	26.38580	0.93320
0.100	2	0.60	20	18	25	18	21	19	0.07032	0.00215	1.72243	3.77092	21.54391	0.76196
0.100	2	0.80	20	18	25	18	21	19	0.07032	0.00287	2.29657	3.77092	18.65758	0.65987
0.100	2	0.90	20	18	25	18	21	19	0.07032	0.00359	2.87071	3.77092	16.68784	0.59021
0.100	2	0.90	20.0	17	25	18	21	19	0.07047	0.00324	2.59299	3.76713	17.55705	0.62101
0.100	2	0.90	20.1	17	25	18	21	19	0.07037	0.00390	3.37124	3.76963	16.01350	0.54463
0.100	2	0.90	20.2	17	25	18	21	19	0.07028	0.00458	4.34975	3.77218	14.78058	0.47948

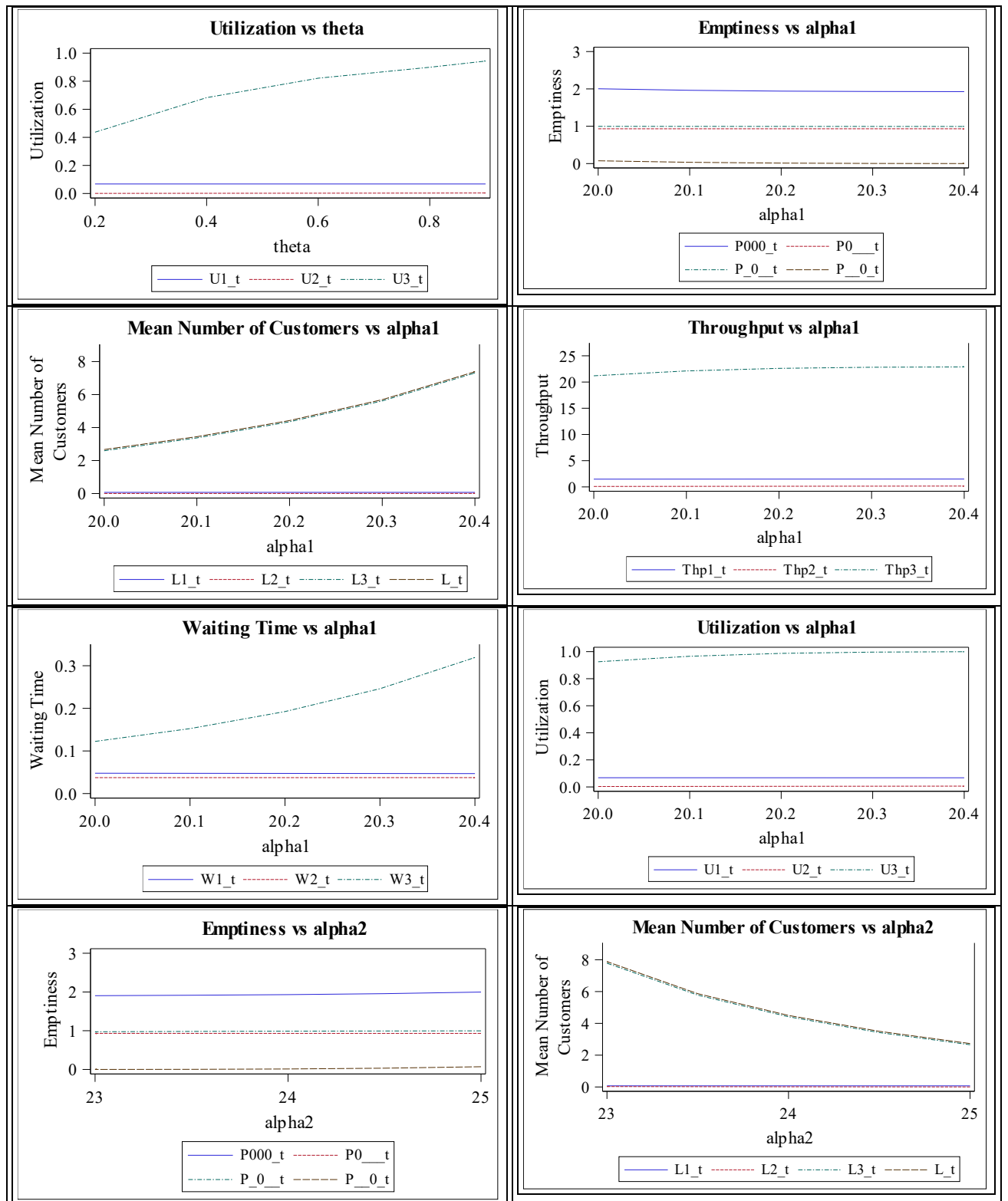
t	λ	θ	α_1	β_1	α_2	β_2	α_3	β_3	V1(t)	V2(t)	V3(t)	CV1(t)	CV2(t)	CV3(t)
0.100	2	0.90	20.3	17	25	18	21	19	0.07018	0.00528	5.61437	3.77478	13.76368	0.42204
0.100	2	0.90	20.4	17	25	18	21	19	0.07008	0.00601	7.30818	3.77742	12.90418	0.36991
0.100	2	0.90	20	16.0	25	18	21	19	0.07061	0.00326	2.60226	3.76334	17.52338	0.61990
0.100	2	0.90	20	15.0	25	18	21	19	0.07075	0.00327	2.61147	3.75955	17.48951	0.61881
0.100	2	0.90	20	14.0	25	18	21	19	0.07089	0.00328	2.62060	3.75576	17.45546	0.61773
0.100	2	0.90	20	12.0	25	18	21	19	0.07118	0.00331	2.63864	3.74819	17.38678	0.61562
0.100	2	0.90	20	11.0	25	18	21	19	0.07132	0.00332	2.64755	3.74441	17.35217	0.61458
0.100	2	0.90	20	10	25.0	18	21	19	0.07147	0.00333	2.65638	3.74063	17.31738	0.61356
0.100	2	0.90	20	10	24.5	18	21	19	0.07147	0.00706	3.42080	3.74063	11.89940	0.54067
0.100	2	0.90	20	10	24.0	18	21	19	0.07147	0.01197	4.41470	3.74063	9.14050	0.47594
0.100	2	0.90	20	10	23.5	18	21	19	0.07147	0.01858	5.77659	3.74063	7.33610	0.41607
0.100	2	0.90	20	10	23.0	18	21	19	0.07147	0.02778	7.78311	3.74063	6.00003	0.35845
0.100	2	0.90	20	10	25	22	21	19	0.07147	0.00313	2.68093	3.74063	17.87338	0.61074
0.100	2	0.90	20	10	25	23	21	19	0.07147	0.00308	2.68707	3.74063	18.01863	0.61004
0.100	2	0.90	20	10	25	24	21	19	0.07147	0.00303	2.69322	3.74063	18.16653	0.60935
0.100	2	0.90	20	10	25	35	21	19	0.07147	0.00250	2.76106	3.74063	19.99624	0.60181
0.100	2	0.90	20	10	25	45	21	19	0.07147	0.00205	2.82317	3.74063	22.08168	0.59516
0.100	2	0.90	20	10	25	18	24	19	0.07147	0.00333	0.28998	3.74063	17.31738	1.85702
0.100	2	0.90	20	10	25	18.2	24	19	0.07147	0.00333	1.11594	3.74063	17.31738	0.94663
0.100	2	0.90	20	10	25	18.4	24	19	0.07147	0.00333	2.57692	3.74063	17.31738	0.62294
0.100	2	0.90	20	10	25	18.6	24	19	0.07147	0.00333	5.61539	3.74063	17.31738	0.42200
0.100	2	0.90	20	10	25	18.7	24	19	0.07147	0.00333	8.70858	3.74063	17.31738	0.33886
0.100	2	0.90	23	16	26	18	25	14	0.06728	0.02121	5.30192	3.85519	6.86713	0.43429
0.100	2	0.90	23	16	26	18	25	15	0.06728	0.02121	5.25137	3.85519	6.86713	0.43638
0.100	2	0.90	23	16	26	18	25	16	0.06728	0.02121	5.20095	3.85519	6.86713	0.43849
0.100	2	0.90	23	16	26	18	25	17	0.06728	0.02121	5.15068	3.85519	6.86713	0.44062

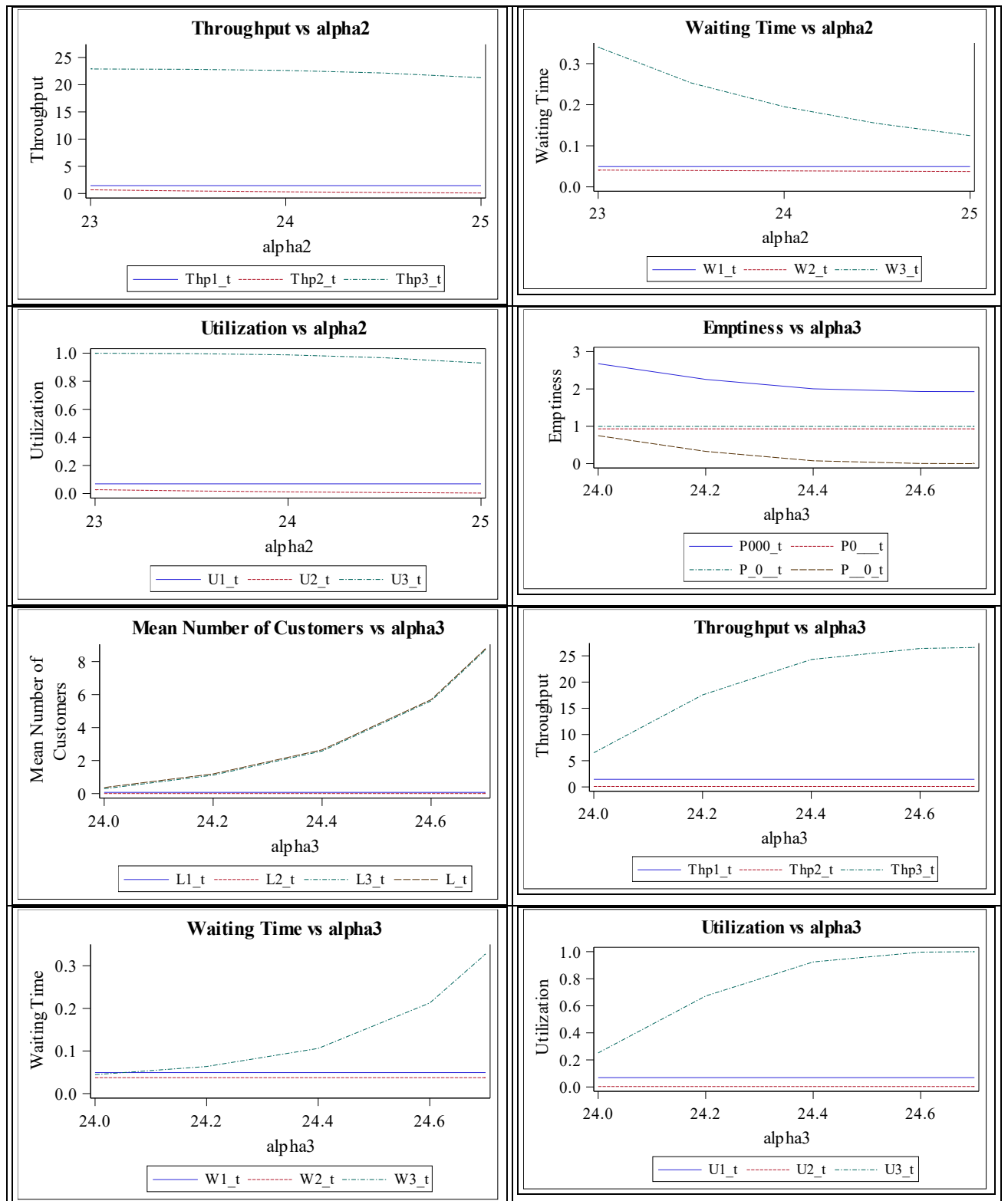
t	λ	θ	α_1	β_1	α_2	β_2	α_3	β_3	V1(t)	V2(t)	V3(t)	CV1(t)	CV2(t)	CV3(t)
0.100	2	0.90	23	16	26	18	25	18	0.06728	0.02121	5.10054	3.85519	6.86713	0.44278
0.100	2	0.90	23	16	26	18	25	22	0.06728	0.02121	4.90141	3.85519	6.86713	0.45169
0.100	2	0.90	24	16	27	19	25	35	0.06602	0.01924	4.81429	3.89182	7.20865	0.45576
0.100	2	0.90	25	17	28	20	26	35	0.06460	0.01745	3.80608	3.93457	7.57042	0.51258
0.100	2	0.90	24	16	27	19	25	70	0.06602	0.01924	3.02609	3.89182	7.20865	0.57486
0.100	2	0.90	24	16	27	19	25	35	0.06602	0.01924	4.81429	3.89182	7.20865	0.45576

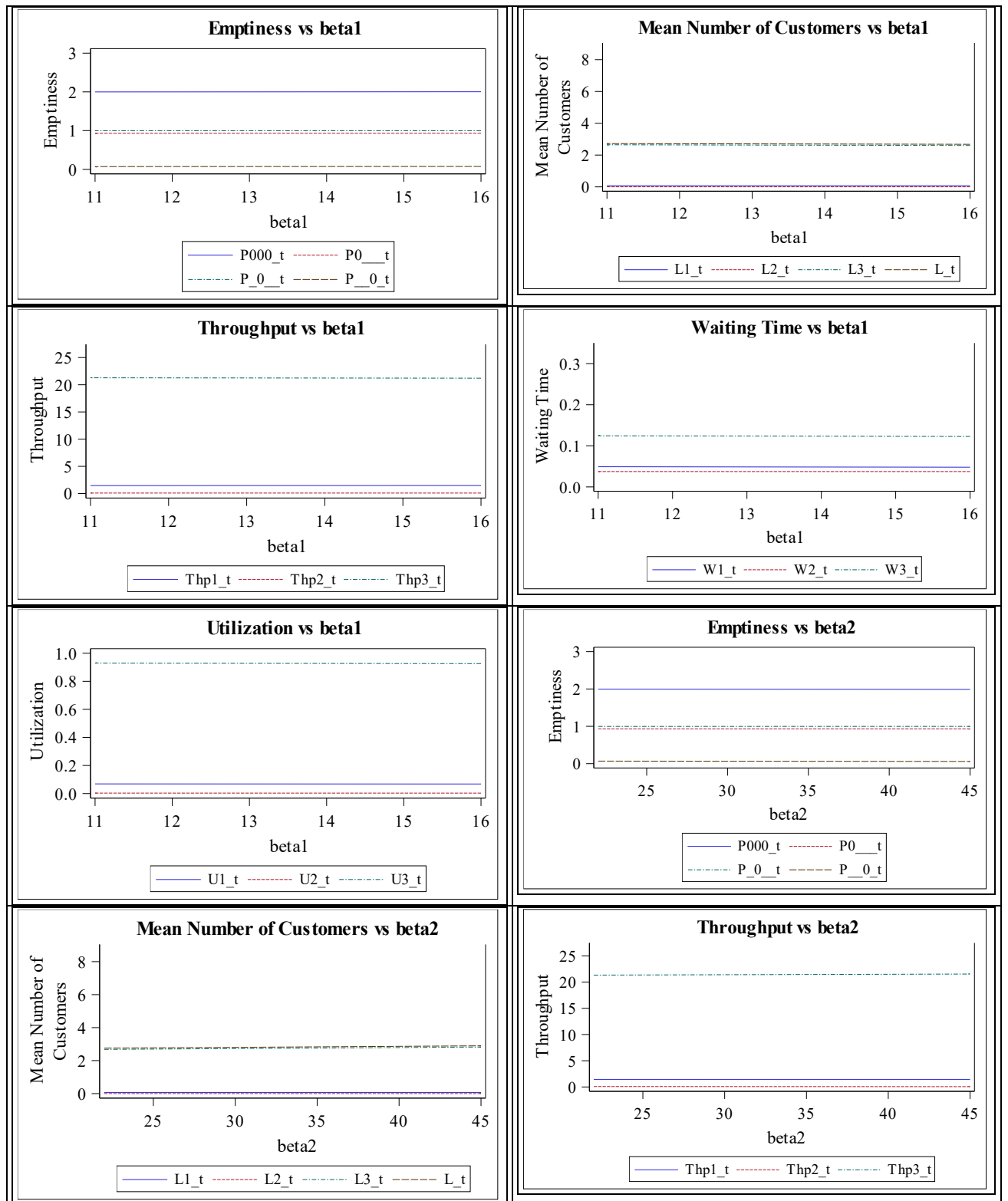
Figure.2 The relationship between the parameters and PMs

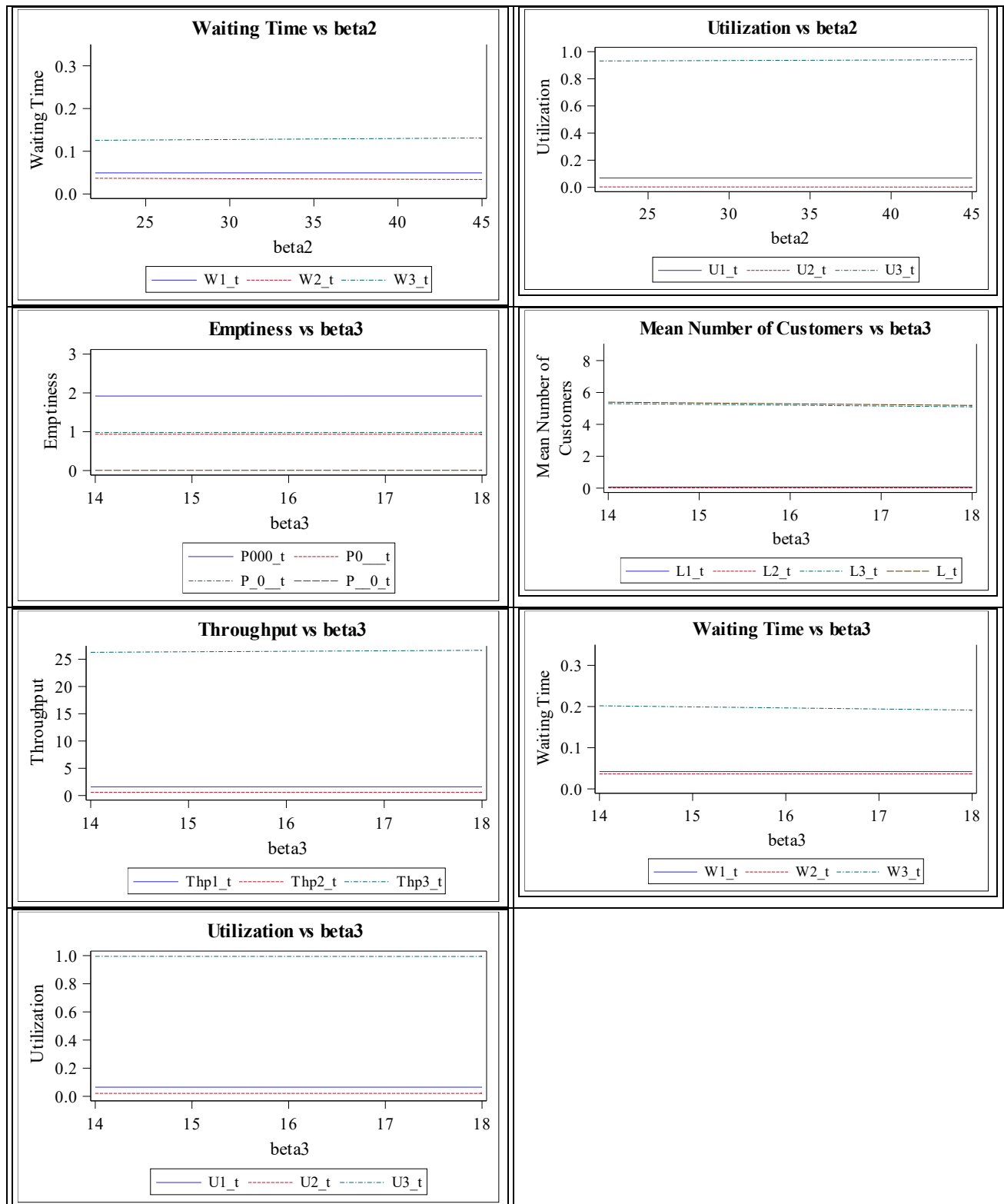












5 Sensitivity analysis:-

This section presents a sensitivity analysis of the proposed three-node tandem queueing system with probabilistic routing. The corresponding results are summarized in Table 4. The baseline parameter values considered for the sensitivity analysis are as follows: $\theta = 1.5$, $t = 0.103$, $\lambda = 2$, $\alpha_1 = 22$, $\beta_1 = 10$, $\alpha_2 = 24.5$, $\beta_2 = 20$, $\alpha_3 = 24$, and $\beta_3 = 35$. The impact of small percentage variations ($\pm 5\%$)

and $\pm 10\%$) in key system parameters such as routing probability θ , time t , arrival rate λ , and service parameters α_i , β_i on various performance measures is examined. The selected base values of parameters are varied systematically, and the resulting changes in mean queue lengths, utilization, throughput, and waiting times are summarized.

Effect of Routing Probability (θ)

As the routing probability θ varies from **-10% to +10%**, the mean number of customers in the first queue remains constant at **0.07050**, while in the second queue it increases from **0.04605 to 0.05628**, and in the third queue it increases significantly from **20.71811 to 25.32213**.

The utilization of the first node remains constant at **0.06807**, whereas the utilization of the second node increases from **0.04500 to 0.05472**, and the third node remains fully utilized.

The throughput of the second node increases from **1.19527 to 1.45349**, while the third node throughput remains constant. The waiting time in the third queue increases from **0.75052 to 0.91730**, indicating increased congestion in downstream nodes.

Effect of Time (t)

As time t varies from **-10% to +10%**, the mean number of customers in the first queue increases from **0.06609 to 0.07392**, while in the second queue it decreases from **0.07129 to 0.03641**, and in the third queue decreases significantly from **28.81440 to 14.32565**.

The utilization of the first node increases from **0.06395 to 0.07125**, while the second node utilization decreases. The throughput of the first node increases from **1.46619 to 1.64823**, while the second node throughput decreases.

The waiting time in the third queue decreases from **1.05762 to 0.51226**, indicating improved system performance over time.

Effect of Arrival Rate (λ)

As the arrival rate λ varies from **-10% to +10%**, the mean number of customers in all queues increases, with $L_1(t)$ increasing from **0.06345 to 0.07755**, $L_2(t)$ from **0.04605 to 0.05628**, and $L_3(t)$ from **20.71811 to 25.32213**.

The utilization of all nodes increases, and the throughput of the first and second nodes increases from **1.41588 to 1.71850** and **1.19527 to 1.45349**, respectively.

The waiting time in the third queue increases from **0.75052 to 0.91730**, showing increased congestion due to higher arrival intensity.

Effect of Service Rate Parameter (α_1)

As α_1 varies from **-10% to +10%**, the mean number of customers in the third queue increases drastically from **5.33938 to 7147.18**, while the second queue also shows a significant increase.

The utilization of the second node increases from **0.00710 to 0.32398**, and the throughput of the second node increases from **0.18866 to 8.60491**.

The waiting time in the third queue increases sharply from **0.19435 to 258.9089**, indicating severe congestion under higher values of α_1 .

Effect of Service Rate Parameter (β_1)

As β_1 varies from **-10% to +10%**, only marginal changes are observed in all performance measures.

The mean queue lengths, utilization, throughput, and waiting times remain almost constant, indicating that the system is relatively insensitive to variations in β_1 .

Effect of Service Rate Parameter (α_2)

As α_2 varies from **-10% to +10%**, the mean number of customers in the second and third queues increases, with $L_3(t)$ increasing from **23.02012 to 28.81440**.

The utilization of the second node increases from **0.04988 to 0.06880**, and the throughput increases from **1.32471 to 1.81325**.

The waiting time in the third queue increases from **0.83391 to 1.05762**, indicating increased load in downstream nodes.

Effect of Service Rate Parameter (β_2)

As β_2 varies from **-10% to +10%**, the mean number of customers in the second and third queues shows moderate variation.

The utilization of the second node decreases initially and then increases, while the throughput follows a similar pattern.

The waiting time in the third queue decreases initially and then increases, indicating a non-linear impact of β_2 .

Effect of Service Rate Parameter (α_3)

As α_3 varies from **-10% to +10%**, the mean number of customers in the third queue decreases from **25.32213 to 10.85644**, while the first and second queues show minor changes.

The utilization of the third node decreases slightly, and the waiting time in the third queue decreases from **0.91730 to 0.39329**, indicating improved service efficiency at the final stage.

Effect of Service Rate Parameter (β_3)

As β_3 varies from **-10% to +10%**, the mean number of customers in the third queue shows extreme variation, increasing up to **7147.18** at certain values.

The utilization of the second node increases significantly, and the waiting time in the third queue increases drastically from **0.83391 to 258.90885**.

This indicates that the system is highly sensitive to variations in β_3 , particularly in downstream performance.

The sensitivity analysis clearly indicates that the routing probability θ and service parameters of downstream nodes have a significant impact on system performance, while parameters like β_1 exhibit relatively lower sensitivity.

Table 4: The values of $L_1(t)$, $L_2(t)$, $L_3(t)$, $U_1(t)$, $U_2(t)$, $U_3(t)$, $Thp_1(t)$, $Thp_2(t)$, $Thp_3(t)$, $W_1(t)$, $W_2(t)$ and $W_3(t)$ for different values of t , λ , α_1 , β_1 , α_2 , β_2 , α_3 , and β_3

Parameter	Selected Value	% change in	PMs											
			$L_1(t)$	$L_2(t)$	$L_3(t)$	$U_1(t)$	$U_2(t)$	$U_3(t)$	$Thp_1(t)$	$Thp_2(t)$	$Thp_3(t)$	$W_1(t)$	$W_2(t)$	$W_3(t)$
θ	1.5	-10%	0.07050	0.04605	20.71811	0.06807	0.04500	0.9999	1.56772	1.56772	1.19527	0.04497	0.03852	0.75052
		-5%	0.07050	0.04860	21.86911	0.06807	0.04744	0.9999	1.56772	1.26007	27.60500	0.04497	0.03857	0.79222
		0%	0.07050	0.05116	23.02012	0.06807	0.04988	0.9999	1.56772	1.32471	27.60500	0.04497	0.03862	0.83391
		5%	0.07050	0.05372	24.17112	0.06807	0.05230	0.9999	1.56772	1.38918	27.60500	0.04497	0.03867	0.87561
		10%	0.07050	0.05628	25.32213	0.06807	0.05472	0.9999	1.56772	1.45349	27.60500	0.04497	0.03872	0.91730
t	0.103	-10%	0.06609	0.07129	28.8144	0.06395	0.0688	0.9999	1.46619	1.81325	27.2445	0.04507	0.03931	1.05762
		-5%	0.06843	0.06045	26.24405	0.06614	0.05866	0.9999	1.5199	1.552	27.42475	0.04502	0.03895	0.95695
		0%	0.07050	0.05116	23.02012	0.06807	0.04988	0.9999	1.56772	1.32471	27.60500	0.04497	0.03862	0.83391

Parameter	Selected Value	% change in	PMs											
			L1(t)	L2(t)	L3(t)	U1(t)	U2(t)	U3(t)	Th p1(t)	Th p2(t)	Thp 3(t)	W1 (t)	W2 (t)	W3(t)
		5 %	0.0 723 2	0.0 432 1	19.07 671	0.0 697 7	0.0 422 9	0.9 999 9	1.6 103 1	1.1 275 4	27.7 852 5	0.0 449 1	0.0 383 2	0.68 658
		10 %	0.0 739 2	0.0 364 1	14.32 565	0.0 712 5	0.0 357 5	0.9 999 9	1.6 482 3	0.9 569 7	27.9 654 8	0.0 448 5	0.0 380 5	0.51 226
		- 10 %	0.0 634 5	0.0 460 5	20.71 811	0.0 614 8	0.0 45	0.9 999 9	1.4 158 8	1.1 952 7	27.6 05	0.0 448 1	0.0 385 2	0.75 052
λ	2	- 5 %	0.0 669 8	0.0 486	21.86 911	0.0 647 8	0.0 474 4	0.9 999 9	1.4 919 3	1.2 600 7	27.6 05	0.0 448 9	0.0 385 7	0.79 222
		0 %	0.0 705 0	0.0 511 6	23.02 012	0.0 680 7	0.0 498 8	0.9 999 9	1.5 677 2	1.3 247 1	27.6 050 0	0.0 449 7	0.0 386 2	0.83 391
		5 %	0.0 740 3	0.0 537 2	24.17 112	0.0 713 5	0.0 523	0.9 999 9	1.6 432 4	1.3 891 8	27.6 05	0.0 450 5	0.0 386 7	0.87 561
		10 %	0.0 775 5	0.0 562 8	25.32 213	0.0 746 2	0.0 547 2	0.9 999 9	1.7 185	1.4 534 9	27.6 05	0.0 451 3	0.0 387 2	0.91 73
		α_1	22	- 10 %	0.0 731 3	0.0 071 3	5.339 38	0.0 705 2	0.0 071	0.9 952 0	1.4 689 4	0.1 886 6	27.4 725 3	0.0 497 8
- 5 %	0.0 718 8			0.0 230 3	10.85 644	0.0 693 6	0.0 227 7	0.9 999 8	1.5 210 8	0.6 047	27.6 044 7	0.0 472 6	0.0 380 9	0.39 329
0 %	0.0 705 0			0.0 511 6	23.02 012	0.0 680 7	0.0 498 8	0.9 999 9	1.5 677 2	1.3 247 1	27.6 050 0	0.0 449 7	0.0 386 2	0.83 391
5 %	0.0 690 3			0.1 207 1	66.29 999	0.0 667	0.1 137 1	0.9 999 9	1.6 094 6	3.0 201 4	27.6 05	0.0 428 9	0.0 399 7	2.40 174
10 %	0.0 678			0.3 915 3	7147. 18	0.0 655 5	0.3 239 8	0.9 999 9	1.6 400 2	8.6 049 1	27.6 05	0.0 413 4	0.0 455	258. 9089
β_1	10	- 10 %	0.0 706 5	0.0 511 8	23.03 889	0.0 682 2	0.0 498 9	0.9 999 9	1.5 639 7	1.3 250 3	27.6 05	0.0 451 8	0.0 386 2	0.83 459
		- 5 %	0.0 705 8	0.0 511 7	23.02 954	0.0 681 4	0.0 498 8	0.9 999 9	1.5 658 5	1.3 248 7	27.6 05	0.0 450 7	0.0 386 2	0.83 425
		0 %	0.0 705 0	0.0 511 6	23.02 012	0.0 680 7	0.0 498 8	0.9 999 9	1.5 677 2	1.3 247 1	27.6 050 0	0.0 449 7	0.0 386 2	0.83 391
		5 %	0.0 705 0	0.0 460 5	20.71 811	0.0 680 7	0.0 450 0	0.9 999 9	1.5 677 2	1.5 677 2	1.19 527	0.0 449 7	0.0 385 2	0.75 052

Parameter	Selected Value	% change in	PMs											
			L1(t)	L2(t)	L3(t)	U1(t)	U2(t)	U3(t)	Th p1(t)	Th p2(t)	Thp 3(t)	W1 (t)	W2 (t)	W3(t)
		10 %	0.0 705 0	0.0 486 0	21.86 911	0.0 680 7	0.0 474 4	0.9 999 9	1.5 677 2	1.2 600 7	27.6 050 0	0.0 449 7	0.0 385 7	0.79 222
α_2	24.5	- 10 %	0.0 705 0	0.0 511 6	23.02 012	0.0 680 7	0.0 498 8	0.9 999 9	1.5 677 2	1.3 247 1	27.6 050 0	0.0 449 7	0.0 386 2	0.83 391
		- 5 %	0.0 705 0	0.0 537 2	24.17 112	0.0 680 7	0.0 523 0	0.9 999 9	1.5 677 2	1.3 891 8	27.6 050 0	0.0 449 7	0.0 386 7	0.87 561
		0 %	0.0 705 0	0.0 562 8	25.32 213	0.0 680 7	0.0 547 2	0.9 999 9	1.5 677 2	1.4 534 9	27.6 050 0	0.0 449 7	0.0 387 2	0.91 730
		5 %	0.0 660 9	0.0 712 9	28.81 440	0.0 639 5	0.0 688 0	0.9 999 9	1.4 661 9	1.8 132 5	27.2 445 0	0.0 450 7	0.0 393 1	1.05 762
		10 %	0.0 684 3	0.0 604 5	26.24 405	0.0 661 4	0.0 586 6	0.9 999 9	1.5 199 0	1.5 520 0	27.4 247 5	0.0 450 2	0.0 389 5	0.95 695
β_2	20	- 10 %	0.0 705 0	0.0 511 6	23.02 012	0.0 680 7	0.0 498 8	0.9 999 9	1.5 677 2	1.3 247 1	27.6 050 0	0.0 449 7	0.0 386 2	0.83 391
		- 5 %	0.0 723 2	0.0 432 1	19.07 671	0.0 697 7	0.0 422 9	0.9 999 9	1.6 103 1	1.1 275 4	27.7 852 5	0.0 449 1	0.0 383 2	0.68 658
		0 %	0.0 739 2	0.0 364 1	14.32 565	0.0 712 5	0.0 357 5	0.9 999 9	1.6 482 3	0.9 569 7	27.9 654 8	0.0 448 5	0.0 380 5	0.51 226
		5 %	0.0 634 5	0.0 460 5	20.71 811	0.0 614 8	0.0 450 0	0.9 999 9	1.4 158 8	1.1 952 7	27.6 050 0	0.0 448 1	0.0 385 2	0.75 052
		10 %	0.0 669 8	0.0 486 0	21.86 911	0.0 647 8	0.0 474 4	0.9 999 9	1.4 919 3	1.2 600 7	27.6 050 0	0.0 448 9	0.0 385 7	0.79 222
α_3	24	- 10 %	0.0 705 0	0.0 511 6	23.02 012	0.0 680 7	0.0 498 8	0.9 999 9	1.5 677 2	1.3 247 1	27.6 050 0	0.0 449 7	0.0 386 2	0.83 391
		- 5 %	0.0 740 3	0.0 537 2	24.17 112	0.0 713 5	0.0 523 0	0.9 999 9	1.6 432 4	1.3 891 8	27.6 050 0	0.0 450 5	0.0 386 7	0.87 561
		0 %	0.0 775 5	0.0 562 8	25.32 213	0.0 746 2	0.0 547 2	0.9 999 9	1.7 185 0	1.4 534 9	27.6 050 0	0.0 451 3	0.0 387 2	0.91 730
		5 %	0.0 731 3	0.0 071 3	5.339 38	0.0 705 2	0.0 071 0	0.9 952 0	1.4 689 4	0.1 886 6	27.4 725 3	0.0 497 8	0.0 377 8	0.19 435
		10 %	0.0 718 8	0.0 230 3	10.85 644	0.0 693 6	0.0 227 7	0.9 999 8	1.5 210 8	0.6 047 0	27.6 044 7	0.0 472 6	0.0 380 9	0.39 329

Parameter	Selected Value	% change in	PMs											
			L1(t)	L2(t)	L3(t)	U1(t)	U2(t)	U3(t)	Th p1(t)	Th p2(t)	Thp 3(t)	W1 (t)	W2 (t)	W3(t)
β_3	35	- 10 %	0.0 705 0	0.0 511 6	23.02 012	0.0 680 7	0.0 498 8	0.9 999 9	1.5 677 2	1.3 247 1	27.6 050 0	0.0 449 7	0.0 386 2	0.83 391
		- 5 %	0.0 690 3	0.1 207 1	66.29 999	0.0 667 0	0.1 137 1	0.9 999 9	1.6 094 6	3.0 201 4	27.6 050 0	0.0 428 9	0.0 399 7	2.40 174
		0 %	0.0 678 0	0.3 915 3	7147. 1800 0	0.0 655 5	0.3 239 8	0.9 999 9	1.6 400 2	8.6 049 1	27.6 050 0	0.0 413 4	0.0 455 0	258. 9088 5
		5 %	0.0 706 5	0.0 511 8	23.03 889	0.0 682 2	0.0 498 9	0.9 999 9	1.5 639 7	1.3 250 3	27.6 050 0	0.0 451 8	0.0 386 2	0.83 459
		10 %	0.0 705 8	0.0 511 7	23.02 954	0.0 681 4	0.0 498 8	0.9 999 9	1.5 658 5	1.3 248 7	27.6 050 0	0.0 450 7	0.0 386 2	0.83 425

6 Comparative study:

In this segment, a comparative analysis is carried out between the proposed three-node tandem queueing model with probabilistic routing and time-dependent (non-homogeneous) service processes, and a model assuming homogeneous arrival and service processes. The comparison is based on key performance measures such as average number of customers in each queue, utilization, throughput, and average waiting time at each service station.

To ensure consistency in evaluation, both models are analyzed under varying values of the time parameter t , specifically for $t=0.100, 0.101, 0.102, 0.103, 0.104$, while keeping other parameters fixed at $\theta=0.9$, $\lambda=2$, $\alpha_1=20$, $\alpha_2=25$, $\alpha_3=21$, $\beta_1=18$, $\beta_2=18$ and $\beta_3=19$. The corresponding performance measures for both models are presented in Table 5.

This comparative study highlights the effect of incorporating probabilistic routing and time-dependent service mechanisms in capturing realistic system dynamics more effectively than the traditional homogeneous model.

At $t=0.100$

At $t=0.100$, the average number of customers in all three queues is lower in the proposed model compared to the homogeneous model. Specifically, Queue 1 shows a reduction of approximately **3.71%**, Queue 2 by **38.89%**, and Queue 3 by **24.45%**.

Similarly, utilization values are lower in the proposed model, particularly in the third node where a reduction of **20.38%** is observed. Throughput of the first node shows an improvement of approximately **4.98%**, while the second and third nodes show variations of **29.00%** and **10.39%**, respectively.

The average waiting time is also significantly reduced, especially in the third queue where a drastic reduction of **326.11%** is observed.

At $t=0.101$

At $t=0.101$, similar trends continue with improved performance in the proposed model. The reduction in queue lengths is more pronounced, with Queue 3 showing a decrease of **28.41%**.

Utilization in the third node decreases by approximately **23.94%**, and throughput improvements are observed in the first node by **4.98%**.

The waiting time in the third queue shows a substantial reduction of **353.96%**, indicating improved system responsiveness.

At $t=0.102$

At $t=0.102$, the benefits of the proposed model become more evident. The third queue shows a reduction in congestion of about **33.42%**, and utilization decreases by **28.46%**.

The throughput difference at the third node increases to **17.61%**, highlighting improved processing capability.

Waiting time in the third queue reduces further by approximately **386.51%**, indicating significant efficiency gains.

At t=0.103

At t=0.103, the proposed model demonstrates substantial improvements. The reduction in the third queue length reaches approximately **39.94%**, while utilization decreases by **34.41%**.

The throughput of the third node shows a notable improvement of **22.95%**, indicating better handling of system load.

The average waiting time in the third queue is reduced by about **425.81%**, emphasizing the effectiveness of the model in reducing delays.

At t=0.104

At t=0.104, the performance gap between the two models becomes even more significant. The third queue shows a congestion reduction of approximately **48.78%**, while utilization decreases by **42.51%**.

The throughput improvement at the third node increases to **30.25%**, demonstrating enhanced system efficiency.

The waiting time in the third queue is reduced by about **472.74%**, indicating substantial improvement in service performance.

Table 5: Comparative study of models with Non-homogeneous and homogeneous PS rates

$\Theta=0.9, \lambda=2, \alpha_1=20, \alpha_2=25, \alpha_3=21, \beta_1=18, \beta_2=18, \beta_3=19$					
t	PMs	Models with Non-Homogeneous arrival and service processes	Models with Homogeneous arrival and service processes	Difference	Percentage of Variation
0.100	L1(t)	0.07032	0.07293	0.00261	3.71160
	L2(t)	0.00036	0.00050	0.00014	38.88889
	L3(t)	0.28707	0.35725	0.07018	24.44700
	U1(t)	0.06791	0.07034	0.00243	3.57827
	U2(t)	0.00036	0.00050	0.00014	38.88889
	U3(t)	0.24954	0.30040	0.05086	20.38150
	Thp1(t)	1.48041	1.40674	0.07367	4.97632
	Thp2(t)	0.00962	0.01241	0.00279	29.00208
	Thp3(t)	5.71451	6.30850	0.59399	10.39442
	W1(t)	0.0475	0.05185	0.00435	9.15789
	W2(t)	0.03732	0.04001	0.00269	7.20793
	W3(t)	0.01329	0.05663	0.04334	326.10986
0.101	L1(t)	0.07079	0.07347	0.00268	3.78585
	L2(t)	0.00032	0.00046	0.00014	43.75000
	L3(t)	0.26286	0.33754	0.07468	28.41056
	U1(t)	0.06834	0.07083	0.00249	3.64355
	U2(t)	0.00032	0.00046	0.00014	43.75000
	U3(t)	0.23115	0.28648	0.05533	23.93684
	Thp1(t)	1.49101	1.41670	0.07431	4.98387

$\Theta=0.9, \lambda=2, \alpha_1=20, \alpha_2=25, \alpha_3=21, \beta_1=18, \beta_2=18, \beta_3=19$					
t	PMs	Models with Non-Homogeneous arrival and service processes	Models with Homogeneous arrival and service processes	Difference	Percentage of Variation
	Thp2(t)	0.00867	0.01145	0.00278	32.06459
	Thp3(t)	5.29777	6.01602	0.71825	13.55759
	W1(t)	0.04748	0.05186	0.00438	9.22494
	W2(t)	0.03729	0.04001	0.00272	7.29418
	W3(t)	0.01236	0.05611	0.04375	353.96440
0.102	L1(t)	0.07124	0.07399	0.00275	3.86019
	L2(t)	0.00029	0.00042	0.00013	44.82759
	L3(t)	0.23754	0.31692	0.07938	33.41753
	U1(t)	0.06876	0.07132	0.00256	3.72309
	U2(t)	0.00029	0.00042	0.00013	44.82759
	U3(t)	0.21143	0.27161	0.06018	28.46332
	Thp1(t)	1.50139	1.42646	0.07493	4.99071
	Thp2(t)	0.00775	0.01052	0.00277	35.74194
	Thp3(t)	4.84985	5.70388	0.85403	17.60941
	W1(t)	0.04745	0.05187	0.00442	9.31507
	W2(t)	0.03727	0.04001	0.00274	7.35176
	W3(t)	0.01142	0.05556	0.04414	386.51489
0.103	L1(t)	0.07167	0.07451	0.00284	3.96261
	L2(t)	0.00026	0.00038	0.00012	46.15385
	L3(t)	0.21107	0.29538	0.08431	39.94409
	U1(t)	0.06917	0.07180	0.00263	3.80223
	U2(t)	0.00026	0.00038	0.00012	46.15385
	U3(t)	0.19028	0.25575	0.06547	34.40719
	Thp1(t)	1.51156	1.43602	0.07554	4.99749
	Thp2(t)	0.00687	0.00962	0.00275	40.02911
	Thp3(t)	4.36828	5.37077	1.00249	22.94931
	W1(t)	0.04742	0.05189	0.00447	9.42640
	W2(t)	0.03724	0.04001	0.00277	7.43824
	W3(t)	0.01046	0.05500	0.04454	425.81262
0.104	L1(t)	0.0721	0.07501	0.00291	4.03606
	L2(t)	0.00022	0.00035	0.00013	59.09091
	L3(t)	0.18342	0.27289	0.08947	48.77876
	U1(t)	0.06956	0.07227	0.00271	3.89592
	U2(t)	0.00022	0.00035	0.00013	59.09091
	U3(t)	0.16758	0.23882	0.07124	42.51104

$\theta=0.9, \lambda=2, \alpha_1=20, \alpha_2=25, \alpha_3=21, \beta_1=18, \beta_2=18, \beta_3=19$					
t	PMs	Models with Non-Homogeneous arrival and service processes	Models with Homogeneous arrival and service processes	Difference	Percentage of Variation
	Thp1(t)	1.52152	1.44539	0.07613	5.00355
	Thp2(t)	0.00603	0.00876	0.00273	45.27363
	Thp3(t)	3.85034	5.01523	1.16489	30.25421
	W1(t)	0.04739	0.05190	0.00451	9.51678
	W2(t)	0.03722	0.04001	0.00279	7.49597
	W3(t)	0.0095	0.05441	0.04491	472.73684

7 Conclusion:

This study presents the design and analysis of a three-node tandem queueing model incorporating probabilistic routing and time-dependent service rates. The model assumes that customers arrive according to a Poisson process and, after receiving service at the first node, proceed to the second node with probability θ or leave the system with probability $(1-\theta)$. The service processes at all three nodes are governed by non-homogeneous Poisson processes with time-dependent service rates. This framework effectively captures the dynamic behaviour of real-world systems where both service capability and routing decisions influence system performance.

Explicit expressions for key performance measures include the average number of customers in each queue, average waiting times, utilization of service stations, throughput, variance, and coefficient of variation have been derived. The numerical analysis demonstrates that variations in system parameters such as time t , arrival rate λ , and routing probability θ have a significant impact on system performance. It is observed that an increase in time improves service efficiency, resulting in reduced congestion in downstream queues. In contrast, higher arrival rates lead to increased queue lengths, delays, and utilization across all service stations.

The introduction of routing probability θ plays a crucial role in controlling the flow of customers within the system. Higher values of θ result in increased load on subsequent nodes, leading to higher congestion, utilization, and waiting times, particularly in the third queue. This highlights the importance of routing decisions in multi-stage service systems.

Sensitivity analysis indicates that the system is highly responsive to variations in routing probability and service parameters of downstream nodes, while certain parameters exhibit relatively moderate influence. The results also show that improper parameter selection may lead to significant congestion buildup in later stages of the system.

Comparative analysis with models assuming homogeneous arrival and service processes reveals that the proposed non-homogeneous model provides improved performance, particularly in terms of reduced queue lengths, lower waiting times, and better adaptability to time-dependent variations. The proposed model more accurately captures transient system behaviour and provides enhanced predictive capability.

In summary, the three-node tandem queueing model with probabilistic routing and time-dependent service rates offers a flexible and realistic framework for analyzing complex service systems. The model is particularly useful for applications in communication networks, healthcare systems, manufacturing environments, and cloud-based services, where both service dynamics and routing mechanisms play a vital role in overall system performance.

The study can be further extended by considering finite capacity systems, priority-based service disciplines, and multi-routing structures to enhance its applicability in more complex real-world scenarios.

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ABBREVIATIONS & NOTATIONS:

Notation	Description
FCFS	First-Come, First-Served
HSR	Homogeneous Service Rate
LANs	Local Arrival Networks
MANs	Metropolitan Arrival Networks
WANs	Wide Arrival Networks
NHPP	Non-Homogeneous Poisson Process
PS	Poisson service
NHSP	Non-homogeneous service processes
HPA	Homogeneous Poisson arrival
Avg.	Average
TQM	Tandem queueing model
PM	Performance measure
NHSR	Non-Homogeneous Service Rate
PGF	Probability Generating Function
t	Time
n_1	Customers in the first queue
n_2	Customers in the second queue
n_3	Customers in the third queue
λ	Arrival rate parameter
θ	Second Queue with probability θ
$1 - \theta$	Leave the systems with probability
α_1	Service Rate Parameter
α_2	Service Rate Parameter
α_3	Service Rate Parameter
β_1	Time dependent service rate parameter
β_2	Time dependent service rate parameter
β_3	Time dependent service rate parameter
no.	Number
$\mu_1(t)$	The service rates at the first service station
$\mu_2(t)$	The service rates at the second service station
$\mu_3(t)$	The service rates at the third service station
$P_{000}(t)$	probability of the queue being empty
$P_{0..}(t)$	initial queue is empty of elements

Notation	Description
$P_{.0}(t)$	Second queue is empty of elements
$P_{..0}(t)$	Third queue is empty of elements
$L_1(t)$	The average number of customers in the initial queue
$L_2(t)$	The average number of customers in the second queue
$L_3(t)$	The average number of customers in the third queue
$U_1(t)$	The Utilization in the initial service station
$U_2(t)$	The Utilization in the second service station
$U_3(t)$	The Utilization in the third service station
$Thp_1(t)$	The throughput in the initial service station
$Thp_2(t)$	The throughput in the second service station
$Thp_3(t)$	The throughput in the third service station
$W_1(t)$	The average waiting time of customers in the initial queue
$W_2(t)$	The average waiting time of customers in the second queue
$W_3(t)$	The average waiting time of customers in the third queue
$V_1(t)$	The variance of the number of customers in the initial queue
$V_2(t)$	The variance of the number of customers in the second queue
$V_3(t)$	The variance of the number of customers in the third queue
Var.	Variance
CV	Coefficient Of Variation
$CV_1(t)$	The Coefficient Of Variation of the number of customers in the initial queue
$CV_2(t)$	The Coefficient Of Variation of the number of customers in the second queue
$CV_3(t)$	The Coefficient Of Variation of the number of customers in the third queue

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