

Soft Graph On An Equivalence Class Of A Minimal Dominating Set Of A Graph

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Abstract:

In this paper, we explore a novel soft graph structure derived from equivalence class of a minimal dominating set in a simple graph. We define a soft graph $G = (G^*, F, K, A)$ where $G^* = (V, E)$ is an undirected graph and $A \subseteq V$ is a minimal dominating set. A mapping $F: A \rightarrow P(V)$ is defined as $F(x) = [x]_A = \{y \in V | (xy) \in E\}$ and a mapping $K: A \rightarrow P(E)$ is defined as $K(x) = \{uv \in E | \{u, v\} \subseteq F(x)\}$ where $x \in A$ and P is a power set. The pair (F, A) is a soft set over V and (K, A) is a soft set over E . Then $H(a) = \langle a \cup F(a) \rangle$ is a subgraph of G^* , $\forall a \in A$.

This framework enables decentralized control, modular subgraph-based planning, dynamic adaptability, and reliable routing, making it well suited for emergency management. The proposed model further supports adaptive reallocation strategies and localized optimization, and can be integrated into AI-driven disaster response systems. Its real-world applicability is illustrated through simulations of relief deployment and coverage in a hypothetical urban network.

1. Introduction:

Graph-theoretic models are widely employed to represent complex systems; nevertheless, classical graph structures often fail to accommodate imprecise or parameter-dependent information. Soft set theory, introduced by Molodtsov in 1999 [1], provides a flexible mathematical framework for handling such uncertainty. Subsequent contributions by Maji et al. [2] developed key operations and applications of soft sets, which later enabled their integration into graph structures.

The concept of soft graphs evolved through the works of M. Gillbert Rani et al. [3], who studied domination in soft graphs, and Rajesh Thumbaraka et al. [4], who formalized soft graph constructions based on soft sets. Further structural developments were presented by K. Palani et al. [5], while Arumugam and Sundarakannan [8] contributed to the theory of equivalence dominating sets.

In this paper, we propose a novel class of soft graphs induced by minimal dominating sets. We rigorously examine their structural properties, characterize their behavior under relevant graph operations, and establish foundational results. Furthermore, we illustrate the applicability of the proposed model through a case study in disaster management, focusing on optimal relief hub placement and efficient urban service coverage.

2. Preliminaries:

This section outlines fundamental definitions and concepts relevant to the study of soft graphs formed using minimal dominating sets and their equivalence classes. All graphs considered are finite, undirected, and simple unless stated otherwise.

Definition 2.1 [1]: Let U and E be an initial universe set and set of parameters respectively. Let $A \subseteq E$. A pair (F, E) is called a soft set over U if and only if F is a mapping of E into the set of all subsets of the set U . i.e., $F: A \rightarrow P(U)$.

Definition 2.2 [4]: Let $G = (V, E)$ be a simple graph, A be any non-empty set. Let R be an arbitrary relation between elements of A and elements of V . That is $R \subseteq A \times V$. A set valued function $F: A \rightarrow P(V)$ can be defined as $F(x) = \{y \in V | xRy\}$. The pair (F, A) is a soft set over V .

Definition 2.3 [3]: A set D is said to be a dominating set of a soft graph (F, A) if for every $x \in A$, a vertex of $F(x)$ in $V - D$ is adjacent to at least one vertex in D .

Definition 2.4 [3]: A dominating set D is called a minimal dominating set if no proper subset of D is a dominating set of G . The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G .

Definition 2.5: Given a dominating vertex $x \in A$, its equivalence class with respect to the relation $R \subseteq A \times V$ is defined as:

$$[x]_A = \{y \in V | (x, y) \in R \text{ or } \{x, y\} \in E\}$$

Definition 2.6 [6]: A 4- tuple $G = (G^*, F, K, A)$ is called a soft graph of G , if it satisfies the following properties:

- i. $G^* = (V, E)$ is a simple graph
- ii. A is a non-empty set of parameters
- iii. (F, A) is a soft set over V
- iv. (K, A) is a soft set over E

$(F(a), K(a))$ is a subgraph of G^* , $\forall a \in A$.

The subgraph $(F(a), K(a))$ is denoted by $H(a)$.

3. Soft graph on an equivalence class of a minimal dominating set

Definition: Let $G^* = (V, E)$ be a simple graph and A be a minimal dominating set. Let subset R of $A \times V$ be an arbitrary relation from A to V . A mapping $F: A \rightarrow P(V)$ can be defined as $F(x) = [x]_A = \{y \in V | (xy) \in E\}$ and a mapping $K: A \rightarrow P(E)$ can be defined as $K(x) = \{uv \in E | \{u, v\} \subseteq F(x)\}$ where $x \in A$ and P is a power set.

A 4- tuple $G = (G^*, F, K, A)$ is called a soft graph on an equivalence class of a minimal dominating set of G , if it satisfies the following properties:

- (i) $G^* = (V, E)$ is a simple graph
- (ii) A is a minimal dominating set
- (iii) (F, A) is a soft set over V
- (iv) (K, A) is a soft set over E
- (v) $H(a) = \langle a \cup F(a) \rangle$ is a subgraph of G^* , $\forall a \in A$.

4. Results and Discussions:

Theorem4.1: For any graph G , $H(x)$ is always a connected graph for $x \in A$.

Proof:

The result follows from the definition of $H(x)$.

Theorem4.2: For any graph G , if A is a singleton set, then $H(x) \cong G$ for $x \in A$.

Proof:

Let G be any graph with n vertices and $A = \{x\}$. Then for each $x_i \in F(x)$, $xx_i \in K(x)$. Also $K(x) = E(G)$. And hence, $H(x) = \langle x \cup F(x) \rangle \cong G$.

Corollary 4.2.1: If $G = K_n$ or W_n or $K_{1,n}$ then $H(x) \cong G$.

Theorem 4.3: For any graph G with $x \in A$, then $d(u, v) \leq 2$ for any $u, v \in V(H(x))$.

Proof:

Let G be any graph with n vertices and $x \in A$. For any $u, v \in F(x)$,

Case(i): Suppose $u = x$, then $xv \in K(x)$. Hence, $d(u, v) = 1$.

Case(ii): Suppose $u \neq x$, there exists a path u, x, v of length 2.

Hence the proof.

Theorem 4.4 For any graph G and $x \in A$, $H(x) = K_{1,p}$ if and only if $\langle F(x) \rangle$ is a null graph, where $p = |F(x)|$.

Proof:

Let G be any graph with n vertices and $x \in A$.

Suppose $H(x) = K_{1,p}$, where $p = |F(x)|$. Then there exists a vertex $y \in V(H(x))$ such that $\deg(y)_{H(x)} = p$ and for any $x_i (\neq y) \in V(H(x))$, $\deg(x_i) = 1$, $1 \leq i \leq n - 1$.

In $H(x)$, for every $x_i \in F(x)$, $xx_i \in K(x)$, $\deg(x)_{H(x)} = p$. We have the following cases:

Case(i): If $y \neq x$, then $y \in F(x)$. Since $|F(x)| = p$ and $\deg(y)_{H(x)} = p$, which is a contradiction.

Case(ii): If $y = x$, then $y \notin F(x)$. Since $H(x) = K_{1,p}$, for any $x_i \in F(x)$, $\deg(x_i) = 1$, $1 \leq i \leq n - 1$. $F(x)$ is independent and hence $\langle F(x) \rangle$ is a null graph.

Conversely,

Suppose $\langle F(x) \rangle$ is a null graph. Then for any $x_i, x_j \in F(x)$, $x_i \notin N(x_j)$ for $1 \leq i \neq j \leq n - 1$ and vice versa.

In $H(x)$, for every $x_i \in F(x)$, $xx_i \in K(x)$, $(1 \leq i \leq n - 1)$, $\deg(x)_{H(x)} = |F(x)|$ and $\deg(x_i)_{H(x)} = 1$.

Therefore, $H(x) = K_{1,p}$.

Corollary 4.4.1: If $G = K_{m,n}$, then $H(x) = K_{1,m}$ or $K_{1,n}$, $\forall x \in A$.

Theorem 4.5: For any graph G and $x \in A$, $H(x)$ is a cyclic graph if and only if $\langle F(x) \rangle$ is not a null graph.

Proof:

Let G be any graph with n vertices and $x \in A$.

Let $H(x)$ is a cyclic graph.

Suppose $\langle F(x) \rangle$ is a null graph, then by theorem 4.4, $H(x) = K_{1,p}$, where $p = |F(x)|$, which is a contradiction. Hence $\langle F(x) \rangle$ is not a null graph.

Conversely,

Suppose $\langle F(x) \rangle$ is not a null graph. Then there exist at least two vertices $x_1, x_2 \in F(x)$ with $x_1x_2 \in K(x)$ and $xx_i \in K(x)$ for each $x_i \in F(x)$, $1 \leq i \leq n - 1$, which forms a closed path x, x_1, x_2, x in $H(x)$. Hence $H(x)$ contains a cycle.

Theorem 4.6: For any graph G and $x \in A$, if there exist at least one vertex $y \in F(x)$ such that $y \notin N(x_i)$ for any $x_i \in F(x)$, $1 \leq i \leq n$, then $H(x)$ contains at least one pendant vertex.

Proof:

Let G be any graph with n vertices and $x \in A$.

Suppose there exists at least one vertex $y \in F(x)$ such that $y \notin N(x_i)$, for any $x_i \in F(x)$, $1 \leq i \leq n$.

Case(i): If $F(x)$ is an independent set, then $\langle F(x) \rangle$ is a null graph. The result follows from theorem4.4.

Case(ii): If $F(x)$ is not an independent set, then for $y \in F(x)$ with $y \notin N(x_i)$ for any $x_i (\neq y) \in F(x)$, $1 \leq i \leq n$, there exists only one path from y to x_i through x . This implies $\deg(y)_{H(x)} = 1$ with $yx \in K(x)$, as a pendant edge. Hence $H(x)$ contains at least one pendant vertex.

Theorem4.7: For any graph G with n vertices and $x \in A$, if $|N(x_i) \cap F(x)| = 2$ for each $x_i \in F(x)$, $1 \leq i \leq n$, then $H(x) = W_n$.

Proof:

Let G be any graph with n vertices and $x \in A$.

Suppose $|N(x_i) \cap F(x)| = 2$ for each $x_i \in F(x)$, $1 \leq i \leq n$. Then $\deg(x_i)_{H(x)} = 3$ so that $H(x) - x$ forms a cycle. And since $N(x) = F(x)$, $H(x) = W_n$, a wheel graph.

Theorem4.8: For any graph G and $x \in A$, $H(x)$ is a Eulerian graph if $\langle F(x) \rangle$ is connected and for each $x_i \in F(x)$, $|N(x_i) \cap F(x)| = 2m - 1$, $1 \leq m \leq n$.

Proof:

Let G be any graph with n vertices and $x \in A$.

Suppose $\langle F(x) \rangle$ is connected and for each $x_i \in F(x)$, $|N(x_i) \cap F(x)| = 2m - 1$, $1 \leq m \leq n$.

In $H(x)$, $|N(x_i) \cap F(x)| \cup \{x\} = (2m - 1) + 1 = 2m$. Then, for each $y \in F(x) \cup \{x\}$, $\deg(y)$ is even. Hence, $H(x)$ is a Eulerian graph.

Theorem4.9: For any graph G and $x \in A$, $H(x)$ is Hamiltonian if one of the following conditions is satisfied:

- (i) For each $x_i \in F(x)$, $|N(x_i) \cap F(x)| = 2$
- (ii) There exists $x_i, x_j \in F(x)$ such that $|N(x_k) \cap F(x)| = 1$ ($k = i, j$) and $|N(y) \cap F(x)| = 2$ for each $y (\neq x_k) \in F(x)$.

Proof:

Let G be any graph with n vertices and $x \in A$.

- (i) Suppose for each $x_i \in F(x)$, $|N(x_i) \cap F(x)| = 2$. Then there exists a cycle connecting all vertices of $F(x)$. By theorem4.7, $H(x)$ is a wheel graph and a Hamiltonian.
- (ii) Suppose there exists $x_i, x_j \in F(x)$ such that $|N(x_k) \cap F(x)| = 1$ ($k = i, j$) and $|N(y) \cap F(x)| = 2$ for each $y (\neq x_k) \in F(x)$. Then $\deg(x_i)_{H(x)-x} = \deg(x_j)_{H(x)-x} = 1$ and $\deg(y)_{H(x)-x} = 2$. This implies that there exists a path connecting all vertices of $F(x)$. Since $N(x) = F(x)$, by the definition, $H(x)$ contains a cycle covering $\{x \cup F(x)\}$. Hence $H(x)$ is Hamiltonian.

Theorem4.10: For any graph G and $x \in A$, if there exists any $x_i, x_j \in F(x)$ such that $x_i x_j \in E(G)$, $1 \leq i \neq j \leq n$, then $H(x)$ is transitive.

Proof:

Let G be any graph with n vertices and $x \in A$.

Suppose there exists any $x_i, x_j \in F(x)$ such that $x_i x_j \in E(G)$, $1 \leq i \neq j \leq n$. Then $x_i x_j \in K(x)$. In $H(x)$, $xx_i, xx_j \in K(x)$. This implies that there is a path from x to x_i , x_i to x_j and x_j to x . Hence $H(x)$ is transitive.

Corollary4.10.1: For any graph G and $x \in A$, $H(x)$ is a graph of an equivalence relation, if there exists any $x_i, x_j \in F(x)$ such that $x_i x_j \in E(G)$, $1 \leq i \neq j \leq n$.

Proof:

Let G be any graph with n vertices and $x \in A$.

Suppose there exists any $x_i, x_j \in F(x)$ such that $x_i x_j \in E(G)$, $1 \leq i \neq j \leq n$. Then Theorem4.10, $H(x)$ is transitive.

Since G is undirected graph, it is reflexive and symmetric. Therefore $H(x)$ is a graph of an equivalence relation.

Theorem4.11: For any graph G with n vertices and $x \in A$, $H(x)$ is a spanning subgraph of G if and only if $H(x) \cong G$.

Proof:

Let G be any graph with n vertices and $x \in A$.

Suppose $H(x)$ is a spanning subgraph of G . Then $F(x) \cup \{x\} = V(G)$.

For any $x_i, x_j \in F(x)$, $xx_i, xx_j \in K(x)$ and if $x_i x_j \in E(G)$, then $x_i x_j \in K(x)$, $1 \leq i \neq j \leq n$. i.e., $K(x) = E(G)$. Hence $H(x) \cong G$.

Conversely, suppose $H(x) \cong G$, the result is obvious.

Theorem4.12: For any graph G , $x \in A$ with $|F(x)| = k$, if for any $x_i \in F(x)$, $|N(x_i) \cap F(x)| \leq 1$, where $1 \leq i \leq n - 1$, then $H(x)$ has at most $\frac{k}{2}$ copies of edge disjoint cycles C_3 at x .

Proof:

Let G be any graph with n vertices and $x \in A$.

Suppose $|F(x)| = k$. $\langle x \cup F(x) \rangle$ has k copies of K_2 .

If for some $x_i \in F(x)$, $|N(x_i) \cap F(x)| = 1$, $1 \leq i \leq n - 1$, then there exists a vertex $x_j \in F(x)$ such that $x_i x_j \in K(x)$. $xx_i x_j$ forms a C_3 . If this is true for every $\frac{k}{2}$ distinct pairs of vertices of $F(x)$, then $H(x) = \langle x \cup F(x) \rangle$ has $\frac{k}{2}$ copies of C_3 sharing a common vertex ' x '.

For some $x_k \in F(x)$, $|N(x_k) \cap F(x)| = 0$, $1 \leq k \neq i \leq n - 1$, then xx_k forms a pendant edge in $H(x)$.

Hence $H(x)$ has at most $\frac{k}{2}$ copies of edge disjoint cycles C_3 at x .

Corollary4.12.1: $H(x)$ is a friendship graph, if for any $x_i \in F(x)$, $|N(x_i) \cap F(x)| = 1$.

Theorem4.13: For any graph G and $x \in A$ with $|F(x)| = k$, if there exists a vertex $y \in F(x)$ such that $N(x) \cap N(y) = F(x) - \{y\}$ and for each $x_i (\neq y) \in F(x)$, $N(x_i) \cap F(x) = y$, then $H(x)$ has $k - 1$ copies of C_3 sharing a common edge.

Proof:

Let G be any graph with n vertices and $x \in A$. Let $|F(x)| = k$.

Suppose there exists a vertex $y \in F(x)$ such that $N(x) \cap N(y) = F(x) - \{y\}$ and for each $x_i (\neq y) \in F(x)$, $N(x_i) \cap F(x) = y$. Then for each $x_i (\neq y) \in F(x)$, $xx_i, x_i y, yx \in K(x)$,

forms a cycle C_3 and $xy \in K(x)$ is the common edge connecting each $x_i (\neq y) \in F(x)$. Since $|F(x) - \{y\}| = k - 1$, there are $k - 1$ copies of C_3 sharing a common edge in $H(x)$.

5. APPLICATIONS:

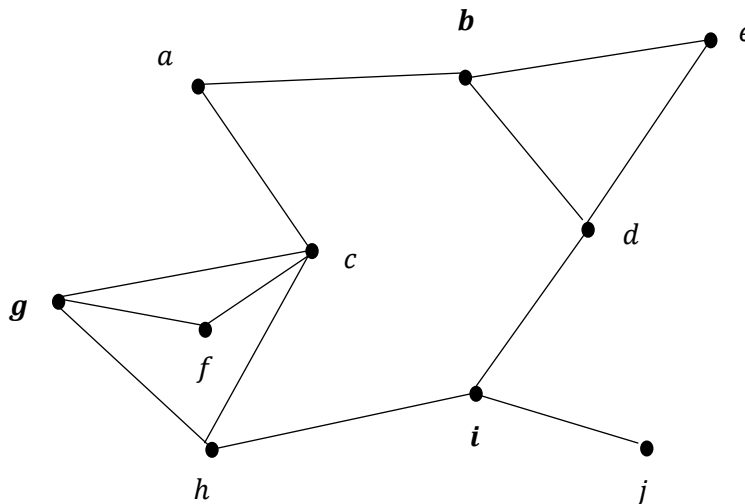
“Simulation of Relief Deployment Using Soft Graphs Formed on Equivalence Classes of Minimal Dominating Sets”

We model an urban network affected by a regional disaster (e.g., earthquake or flood), where locations (nodes) are to be served by decentralized relief hubs. The city's infrastructure is represented as a simple graph $G^* = (V, E)$.

Let $V = \{a, b, c, d, e, f, g, h, i, j\}$ (locations or facilities),

$E = \{ab, ac, bd, be, cf, cg, ch, de, fg, gh, hi, id, ij\}$ (travel paths/roads).

Figure1



We choose $A = \{b, g, i\}$ as a minimal dominating set which represents relief hubs.

Let b = Supply center for north-east zones, g = Food/ Water logistics unit, i = Emergency medical base.

For $b \in A, F(b) = \{a, d, e\}$ and $K(b) = \{ab, bd, be, de\}$

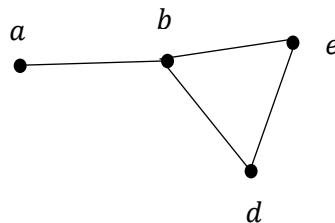


Figure2. $H(b)$

In figure2, the hub **b** is a central logistics hub serving upper-north areas: *a*, *d*, and *e*. Relief trucks move along *ab*, *bd* and *be*, while *de* allows lateral coordination between *d* and *e*.

For $g \in A$, $F(g) = \{c, f, h\}$ and $K(g) = \{cg, cf, ch, fg, gh\}$.

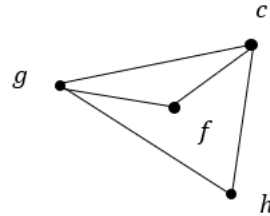


Figure3. $H(g)$

In figure3, the hub **g** is a field base for distributing food and water in central-west areas. It efficiently connects to *c*, *f* and *h*, allowing mobile units to rotate between neighbourhood clusters.

For $i \in A$, $F(i) = \{d, h, j\}$ and $K(i) = \{di, hi, ij\}$

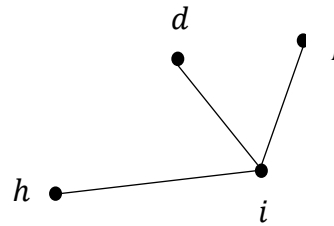


Figure4. $H(i)$

In figure3, the hub **i** manages emergency response in the southern corridor (*d*, *h* and *j*). It forms a linear relief corridor towards the city's edge.

From figures 1, 2 and 3, we observe that, the road *ch* enhances *g*'s internal mesh, making zone *g* more robust to failure. The road *di* bridges *b*'s and *i*'s zones, enabling inter-hub coordination during emergency. And nodes *h* and *d* are shared, offering fault tolerance and flexible reallocation.

Here, each relief hub operates autonomously within its designated subgraph, enabling efficient, localized disaster response. The minimal dominating set ensures complete urban coverage, with every node either being a hub or adjacent to one. Shared nodes between hubs facilitate coordination, enabling resource sharing and operational resilience. This structure balances independence with strategic inter-hub cooperation.

Research Benefits of The Model

Feature	Practical Impact
Decentralized Relief Planning	Subgraphs $H(x)$ allow independent operations under local control
Edge-Based Awareness (K)	Ensures relief roads are clearly identified for each region

Adaptability	If a hub fails (e.g., g , b or i) can dynamically redefine f to cover affected zones
Reduction of Redundancy and Overlap	Node h is shared between g and i which allows backup and handoff if one fails. Similarly, d .
Scalability	Soft set mappings make it easy to scale or reassign regions in real-time systems

Conclusion:

In this paper, a new class of soft graphs are constructed using minimal dominating sets and an associated equivalence relation over the vertex set. By defining mappings from the dominating set to vertex and edge power sets, we established a structured framework to analyse local subgraphs as equivalence classes. The concept was then applied to simulate relief hub deployment and urban coverage, demonstrating how each autonomous hub ensures full coverage through cooperative overlap. This model provides a novel approach for uncertainty management and decision-making in complex networks such as disaster response systems. Future research may explore algorithmic strategies for optimizing such soft graph structures and extend the model to dynamic or weighted networks.

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