Topological Indices On Two Fold Wheel Graphs

S. Kavithaa[1*] , T.S. Thilagavathi [2] , R. Anbu[3] , R. Priya[4]

[1*]Associate Professor , Department of Mathematics, Science and Humanities, Roever Engineering College, Perambalur-621212, Tamil nadu, India.
[2]Assistant Professor , Department of Mathematics, Science and Humanities, Roever Engineering College, Perambalur-621212, Tamil nadu, India.
[3]Assistant Professor, Department of Mathematics, Science and Humanities, Sri Sairam Engineering College, Chennai.
[4]Research Scholar(PT), Department of Mathematics, School of Engineering and Technology, Dhanalakshmi Srinivasan University, Perambalur, and Assistant Professor of Mathematics, Roever Engineering College, Perambalur-621212, Tamil nadu, India.

Abstract:
Graph theory is a lively research field with solid theory and many practical uses, like in medicine and communication. Key ideas such as distance and detour distance between vertices are useful for modeling real-world problems. These concepts have given rise to several important topological measures, including the Wiener Index and the Detour Index. From these, the Reverse Wiener Index and Reverse Detour Index are further developed by making use of a graph's diameter and detour diameter.
In this paper, we introduce the notion of the Wiener Detour Distance Matrix**.** Based on this matrix, we define a family of reverse indices, namely the Reverse Harary Wiener Index**,** Reverse Harary Detour Index**,** Reverse Harary Circular Index, all derived using the diameter and detour of a graph. Finally, these indices are explicitly calculated and examined for the two-fold wheel graph $W_{2,n}$.

Keywords: Wiener index, Detour index, Two fold Wheel graphs, Reverse Wiener index, Reverse detour index, Reverse Harary Wiener index, Reverse Harary Detour index, Reverse Harary Circular index.

1 Introduction

The Topological index is a number associated with a chemical structure represented by a connected graph wherein atoms are represented by vertices and covalent bonds by edges connecting adjacent vertices. Such a graph is known as molecular graph[1]. The graph invariants are usually related to the distance function d(u,v). The first topological index of this type was proposed in 1947 by the chemist Harold Wiener as an aid to determining the boiling point of Paraffin [11]. It is defined as the sum of all distances d(u,v) between the vertices u,v of a graph under consideration[2]. Another topological index, called the detour index, is conceptually close to Wiener index, except that its definition refers to the longest distance D(u,v) instead of the shortest distance between the vertices u,v of a graph[15][4]. That is the detour index is the sum of the longest distance between all pairs of vertices of the graph and the circular index is the sum of D(u,v) & d(u,v). From these indices Reverse Wiener Index and Reverse Detour Index are defined using the diameter and Detour diameter of a graph.
The aim of this paper is to compute the Reverse Harary wiener index, Reverse Harary Detour index, Reverse Harary Circular index[6,9] are defined using the diameter and Detour diameter of a graph and the above indices have been estimated for the Two fold wheel graph $W_{2,n}$.

2 Definitions

Let G be a finite simple connected graph with the vertex set V(G) and the edge set E(G) respectively. In this section the necessary definitions are given below which are used for the main results.

Definition 2.1:
Let d(u,v) denotes the shortest distance between two vertices u,v ∈ V(G). The Wiener polynomial [2] of a graph G with n vertices and m edges in the variable x is denoted by WP(G:x) and is defined as $WP(G:x) = \sum_{u,v \in V(G)} x^{d(u,v)}$. The Wiener index of G is WI = ΣWP′(G:1) where '′' denotes the derivative of WP(G:x) with respect to 'x'.

Definition 2.2:
Let D(u,v) denotes the longest distance between two vertices u,v ∈ V(G). The Detour distance polynomial [17] of a graph G is denoted by $DP(G:x) = \sum_{u,v \in V(G)} x^{D(u,v)}$. The Detour index is given by DI = ΣDP′(G:1) where '′' denotes the derivative of DP(G:x) with respect to 'x'.

Definition 2.3:
Let $d^o(u, v)$ denotes the circular distance [16]between two vertices u,v∈V(G). The circular polynomial of a graph G is denoted by CP(G:x) and is defined as CP(G:x) = D(u,v)+d(u,v). The circular index of G is CI = CP′(G:x), where '`' denotes the derivative of CP(G:x) with respect to 'x'.

Definition 2.4:
The Wiener Matrix (WM)[3] is a lower (or) upper triangular matrix whose elements are Wiener Distances d(u,v).

Definition 2.5:
The Detour Matrix (DM)[5] is a lower (or) upper triangular matrix whose elements are Detour Distances D(u,v).

Definition 2.6:
The Wiener Detour Matrix is a vertex set to vertex set square matrix, whose elements are zero's along the principal diagonal, the shortest distance d(u,v) below the principal diagonal and the Detour distance D(u,v) above the principal diagonal and it is denoted by WDM.

Definition 2.7:
A circular matrix(CM) of G is a lower or upper triangular matrix whose elements are $d^o(u,v)$ [16].

Definition 2.8:
The eccentricity $e_d(u)$ of a vertex 'u' is the distance to a vertex farthest from u.

Definition 2.9:
The radius r(G) of G is defined by $r(G) = \min\{e_d(u) : u \in V(G)\}$, and the diameter, d(G) of G is defined by $d(G) = \max\{e_d(u) : u \in V(G)\}$.

Definition 2.10:
The detour eccentricity $e_D(u)$ of a vertex 'u' is the distance to a vertex farthest from u [17,18].

Definition 2.11:

The detour radius R(G) is defined by R(G) = min{$e_D(u)$: u ∈ V(G)}, and the detour diameter, D(G) of G is defined by D(G) = max{$e_D(u)$: u ∈ V(G)}.

Definition 2.12:

The reverse Wiener Index of a graph G [8] denoted by $\wedge_d(G)$ is defined as $\wedge_d(G) = \sum_{i<j} r_{ij}$ where r_{ij} is obtained as $r_{ij} = \begin{cases} d - d_{ij} & \text{if } i \neq j \\ 0, & \text{otherwise} \end{cases}$ where d is the diameter of the graph 'G' and d_{ij} is the length of the shortest path between vertices v_i and v_j.

Definition 2.13: The reverse Detour Index of a graph G denoted by $\wedge_D(G)$ is defined as $\wedge_D(G) = \sum_{i<j} R_{ij}$ where R_{ij} is obtained as $R_{ij} = \begin{cases} D_{ij} - d_D & \text{if } i \neq j \\ 0, & \text{otherwise} \end{cases}$ where D is the Detour diameter of the graph 'G' and D_{ij} is the length of the longest path between the vertices v_i and v_j .

Definition 2.14*:*
The reverse circular index of a graph G [7] is denoted by $\wedge_c(G)$ is defined as, $\wedge_c(G) = \sum_{i<j} (r_{ij} + R_{ij})$

Definition 2.15:
Let d(u,v) denotes the shortest distance between two vertices u,v ∈ V(G). The Harary Wiener polynomial of a graph G is denoted by HWP(G:x) and is defined as HWP(G:x) = $\sum_{u,v \in V(G)} x^{\frac{1}{(d(u,v))^2}}$ .The Harary Wiener index of G is HWI = ΣHWP′(G:1) where '′' denotes the derivative of HWP(G:x) with respect to 'x'. And Harary Detour index of G is HDI = ΣHDP′(G:1) where HDP(G:x) = $\sum_{u,v \in V(G)} x^{\frac{1}{(D(u,v))^2}}$

Definition 2.16:
Let G is the graph derived from a wheel by duplicating the hub vertex one or more times. We call G a t-fold wheel if there are t hub vertices, each adjacent to all rim vertices and not adjacent to each other.
A two fold wheel is shown in Fig 20.

3. MAIN RESULTS

3.1. TOPOLOGICAL INDICES OF TWO FOLD WHEEL GRAPH:

***THEOREM 3.1.1*:**The Reverse wiener index, Reverse detour index and Reverse circular index of Two fold wheel graph $W_{2,n}$ are respectively. if n ≥ 3

$$\Lambda(W_{2,3}) = 3n \quad ; \quad \Lambda_D(W_{2,3}) = 0 \quad ; \quad \Lambda_C(W_{2,3}) = 3n$$

Proof:
Let $V(W_{2,n})=\{u_i/1\leq i\leq n+2\}$ and $E(W_{2,n})=\{u_iu_{i+1}/1\leq i\leq n-1\} \cup \{u_nu_1\} \cup \{u_{n+1}u_i/1\leq i\leq n\} \cup \{u_{n+2}u_i/1\leq i\leq n\}$ be the vertex set and edge set of $W_{2,3}$ respectively.

The Wiener-Detour matrix of $W_{2,3}$ are respectively given in Fig:21

Fig :21-WDM($W_{2,3}$)

	u_1	u_2	u_3	u_4	u_5
u_1	**0**	**4**	**4**	**4**	**4**
u_2	**1**	**0**	**4**	**4**	**4**
u_3	**1**	**1**	**0**	**4**	**4**
u_4	**1**	**1**	**1**	**0**	**4**
u_5	**1**	**1**	**1**	**2**	**0**

Using the distance matrices in Fig:21 , we computed the following polynomials and indices.

$WP(W_{2,3}:x)=9x+1x^2$ $WP'(W_{2,3}:1)=11$

$DP(W_{2,3}:x)=10x^4$ $DP'(W_{2,3}:1)=40$

$CP(W_{2,3}:x)=9x^5+1x^6$ $CP'(W_{2,3}:1)=51$

In general, when n≥3 ,

$$WI(W_{2,3}) = n^2+2 \quad ;$$

$$DI(W_{2,3}) = \frac{n^3+4n^2+5n+2}{2} \quad ;$$

$$CI(W_{2,3}) = \frac{n^3+6n^2+5n+6}{2}$$

And The Reverse Wiener index, Reverse Detour index, Reverse Circular index are calculated from the Reverse wiener detour matrices & Reverse circular matrices .

Fig:22-RWDM($W_{2,3}$)

	u_1	u_2	u_3	u_4	u_5
u_1	**0**	**0**	**0**	**0**	**0**
u_2	**1**	**0**	**0**	**0**	**0**
u_3	**1**	**1**	**0**	**0**	**0**
u_4	**1**	**1**	**1**	**0**	**0**
u_5	**1**	**1**	**1**	**0**	**0**

$RWP(W_{2,3}:x)=9x+1x^0$ $RDP(W_{2,3}:x)=10x^0$

RWP′($W_{2,3}$:1) = 9 RDP′($W_{2,3}$:1) = 0 RCP′($W_{2,3}$:1) = 9
In general, when n≥3, we get

$\Lambda(W_{2,3}) = 3n$; $\Lambda_D(W_{2,3}) = 0$; $\Lambda_C(W_{2,3}) = 3n$

THEOREM 3.1.2: The Reverse Harary wiener index, Reverse Harary detour index and Reverse Harary circular index of Two fold wheel graph $W_{2,n}$ are respectively. if n ≥ 3

$$\Lambda HW(W_{2,3}) = \frac{7n^2 + 3n + 14}{8}$$

$$\Lambda_D HD(W_{2,3}) = \frac{-n^4 - 5n^3 - 9n^2 - 6n}{2(n+1)}$$

$$\Lambda_C HC(W_{2,3}) = \frac{-4n^4 - 13n^3 - 26n^2 - 7n + 14}{8(n+1)}$$

Proof: Using the above distance matrices (Fig-21) , we can generate the Harary wiener detour matrices & Reverse Harary Wiener Detour matrices. From that matrix we can calculate the following terms $WP(W_{2,3}:x)=9x+1x^{1/4}$ WP′($W_{2,3}$:1) =37/4
$DP(W_{2,3}:x) = 10x^{1/16}$ DP′($W_{2,3}$:1) = 5/8
CP′($W_{2,3}$:1) =79/8
In general, when n≥3

$$\Lambda HW(C_3P_n) = \frac{2227n^2 - 4183n + 1956}{288}$$

$$\Lambda_D HD(C_3P_n) = \frac{3n^7 + 3n^6 - 14n^5 - 3n^4 + 7n^3 + 6n^2 - 8n + 2}{2n(n-1)(n+1)^2}$$

$$\Lambda_C HC(C_3P_n) = \frac{432n^7 + 2659n^6 - 3972n^5 - 4886n^4 + 4920n^3 + 3091n^2 - 3108n + 288}{288n(n-1)(n+1)^2}$$

$RHWP(W_{2,3}:x)=9x+1x^{7/4}$; RHWP′($W_{2,3}$:1) =43/4
$RHDP(W_{2,3}:x) = 10x^{-63/16}$; RHDP′($W_{2,3}$:1) = -315/8 ; RCP′($W_{2,3}$:1) = -229/8
In general, when

$$\Lambda HW(W_{2,3}) = \frac{7n^2 + 3n + 14}{8}$$

$$\Lambda_D HD(W_{2,3}) = \frac{-n^4 - 5n^3 - 9n^2 - 6n}{2(n+1)}$$

$$\Lambda_C HC(W_{2,3}) = \frac{-4n^4 - 13n^3 - 26n^2 - 7n + 14}{8(n+1)}$$

4. Communication Network Design

Two fold wheel graphs are very useful for satellite communication system simulate, two circular client layers linked by central servers, and used in a backup ring in hub and spoke systems. Degree-based indices are used to quantify the hub node's load importance and also used in identify the last minute rises. Eccentricity based indices calculate the more delay in the bad case communication , concentrate reliability assessment but distance based indices help average communication delay and optimize the routing. Due to these, two fold wheel graphs are helpful in 5G small cell architecturers model, IoT based network systems and topology design for data center.

5. Transportation and Urban planning

A two-fold wheel graph helps to show a city center connected with inner and outer ring roads and circular lines connected to a central interchange in metro networks in transportation systems. Generally the Wiener index calculates average travel distance, the Harary index computes the overall accessibility, and the Zagreb-related indices help to compute the traffic load distribution. These indices help to support smart city traffic planning, optimize the route by emergence, and congestion analysis by evaluating connectivity and efficient travel.

6. Power Grid and Electrical Networks:

Two fold wheel graphs are more useful to model the substation network and clusters of smart grid micro-grid. Degree based indices in these system helps to compute robustness of system and mention the vulternable nodes. It also helps to tolerate the the fault. By distance and connectivity criteria, engineers can make suitable design of more resilient and constant power distribution systems.

7. Computer Science and Data Structures:

Two fold wheel graphs present in computing system, architecture with blockhain layered, interconnection networks with parallel processor in computer science. The Spectral radius compute to evaluate processor load with degree central, quantify the local redundancy with clustering coefficients, execution of multiple efficiency. These indices also perform optimizing work, confirm about balanced processing, and reliability improvement.

8. Importance of Two-fold wheel graphs:

Higher connectivity provided by tow-fold wheel graphs comparing with simple wheel graphs. Fault tolerance also better comparing with simple wheel graphs. Path length average also lower, so transportation cost will be minimized. Load distribution would be balance. These advantages very helpful for modelling resilient real-world system around engineering and applied sciences.

9. Research Significance:

Closed form expression, designing the optimal network topologies, graph comparison with other, predictive models in chemical improvement, computational applications are the research areas by the study of topological indices on two-fold wheel graphs.

References

[1] Baskar Babujee.J and Ramakrishnan.S, “Topological Indices for graphs and chemical reactions”, ICMCS 2011, PP.81-88.

[2] Joshi.A and Baskar Babujee.J,” Wiener polynomial for graphs with cycles”,ICMCS 2008, PP.119-225.

[3] Kaladevi.V, Anuradha.R and Selvarani.P,” Two polynomials in one matrix”,ICMEB-2012, ISBN: 978-81-8286-015-5; PP.133-135.

[4] Kaladevi.V and Backialakshmi.P,”Detour distance polynomial of double star graph and Cartesian product of P2 and Cn”, Antarctica Journal of Mathematics, Volume 8(2011).

[5] Diudea, M. V , “Wiener and hyper – Wiener numbers in a single matrix”. J. Chem. Inf. Comput. Sci. 1996,36,833-836.

[6] Gordon Cash ; Sandi Klavzar ; Marko Petkovsek; Three Methods for calculation of the hyper – Wiener index of molecular graphs.J. Chem. Inf. Comput. Sci.2002 , 42 , 571-576

[7] Kaladevi.V and Kavithaa.S, Reverse Circular index of some graphs, NSAGT 2012,PP.78-83
[8] P. Dankelmann,I. Gutman, S. Mukwembi, H.C. Swart, "The Edge Wiener index of a graph, journal Discrete Mathematics 309 (2009) 3452_3457
[9] Kaladevi.V and Kavithaa.S,,"Fifteen Reverse topological indices of a graph in a single distance matrix", Jamal Academic Research Journal-2014,ISSN:0973- 0303,PP.1-9
[10] Kaladevi.V and Kavithaa.S, On Varieties of Reverse Wiener like indices of a graph, International Journal of Fuzzy mathematics Archive, ISSN: 2320 –3242 (P), 2320 – 3250 (online),Vol.4. No.1.pp 37-46
[11] H. Wiener, Structural determination of Paraffin boiling points, J. Amer. Chem. Soc. 69 (1947) 17-20.
[12] I. Gutman, Y.N. Yeh, S.L. Lee, Y.L.Luo, Some recent results in the theory of the Wiener Number, Indian Journal of Chemistry, 32A (1993) 651-661
[13] I. Gutman, S.Klavzar, B. Mohar (Eds), Special issue on the Wiener index, Discrete Applied Mathematics 80 (1997), pp.1-115.
[14] A.T. Balaban, Topological indices based on topological distances in molecular graphs, Pure & Appl. Chem. 55 (1983),199-206
[15] Santhakumaran.A.P, Detour number of a graph, NCGTGA-2012
[16] G.Chartrand, H. Escuadro and P.Zang, "Detour distance graphs, J. Combin. Math. Combin. Comput. Volume 53(2005), pp,75-94
[17] A.Mohammed Yusuf Shadik and G.Marimuthu, "The Circular distance in graphs", NSRTDG-2012
[18] S. Kavithaa, R. Anuradha, S. Dineshkumar, R. Uma, R. Suganya, "Detour Padmakar ivan indices of Harmonic graphs, International journal of applied mathematics, Aug 2025 / 38 No. 7s / 1133-1144.